Designing right endpoint boundary feedback stabilizers for the linearized Korteweg-de Vries equation using left endpoint boundary measurements

AHMET BATAL* & TÜRKER ÖZSARI

Department of Mathematics, Izmir Institute of Technology
Urla, Izmir, TURKEY

Abstract. In this paper, we design an observer for the linearized Korteweg-de Vries (KdV) equation posed on a finite domain. We assume that there is a sensor at the left end point of the domain capable of measuring the first and second order boundary traces of the solution. Using only the partial information available, we construct Dirichlet-Neumann boundary controllers for the original system acting at the right endpoint so that the system becomes exponentially stable. Stabilization of the original system is proved in the mean-square sense, while the convergence of the observer system to the original plant is proved also in higher order Sobolev norms. The standard backstepping approach used to construct a left endpoint controller fails and presents mathematical mathematical challenges when building right endpoint controllers, due to the overdetermined nature of the related kernel models. In order to deal with this difficulty we use the *pseudo-backstepping* method that we recently introduced in (Özsarı & Batal, 2018). This technique is based on using an imperfect kernel and still provides an exponentially stabilizing controller, although with the cost of a low exponential rate of decay. At the end of the paper, we give numerical simulations illustrating the efficacy of our controllers.

Contents

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Key words and phrases. Korteweg-de-Vries equation; back-stepping; pseudo-backstepping; feedback stabilization; and boundary controller.

*Correspondence: Ahmet Batal, Department of Mathematics, Izmir Institute of Technology, Urla, Izmir 35430, TURKEY; E-mail: ahmetbatal@iyte.edu.tr.

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1. Introduction

In this paper, we study the boundary feedback stabilization of the linearized Korteweg-de Vries (KdV) equation on a bounded domain $\Omega = (0, L) \subset \mathbb{R}$. The linearized version of the model under consideration is given by

$$\begin{cases} u_t + u_x + u_{xxx} = 0 \text{ in } \Omega \times (0, T), \\ u(0,t) = 0, u(L,t) = U(t), u_x(L,t) = V(t) \text{ in } (0,T), \\ u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

whereas the nonlinear version of this model is written with the main equation replaced by

$$u_t + u_x + u_{xxx} + uu_x = 0. \tag{1.2}$$

In (1.2), $u = u(x,t)$ can for example model the evolution of the amplitude of a weakly nonlinear shallow dispersive wave (Korteweg & de Vries, 1895). The inputs $U(t)$ and $V(t)$ at the right end point of the boundary are feedback controllers. (1.1) and (1.2) with homogeneous boundary conditions ($U = V \equiv 0$) are both dissipative, since $\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \leq 0$. However, this does not always guarantee exponential decay. It is well-known that if $L \in \left\{2\pi\sqrt{\frac{k^2+kl+l^2}{3}}, k, l \in \mathbb{N}\right\}$ (so called *critical lengths* for KdV), then the solution does not need to decay to zero at all. For example if $L = 2\pi$, $u = 1 - \cos(x)$ is a (time independent) solution of (1.1) on $\Omega = (0, 2\pi)$, but its L^2-norm is constant in t.

If the system is fully observable, one can attempt to construct exponentially stabilizing backstepping controllers for (1.1) and (1.2). However, such an attempt to control from the right endpoint, when only one boundary condition is specified at the left, brings serious mathematical challenges since the related kernel models become overdetermined (see for instance (Cerpa & Coron, 2013) for a discussion of this issue). This problem is not present if one controls the system from the left endpoint (Cerpa & Coron, 2013) or alternatively controls from the right with two boundary conditions specified at the left instead (Tang & Krstic, 2013a). In order to deal with the difficulty associated with backstepping from the right, we recently introduced the *pseudo-backstepping* method (Özsarı & Batal, 2018). This method is based on constructing a backstepping controller which uses only an imperfect kernel. We have shown that exponential stability can still be achieved by using a pseudo-backstepping controller with the penalty of a low exponential rate of decay.

On the other hand, if the system is not fully observable such as when there is no full access to the medium, one cannot use a controller that requires measurement of the full state of the original system. In such a case, one generally constructs an observer system if some partial information such as a boundary measurement is available. In this paper, we will assume that there is a sensor at the left end point of the channel capable of measuring the boundary traces $u_x(0,t)$ and $u_{xx}(0,t)$. Our purpose is to construct right endpoint feedback stabilizers for the original system using the output of a suitably chosen observer system that only uses the partial information extracted from the boundary. From the mathematical point of view, the question is as follows:

Problem 1.3. *Can you write an initial boundary value problem (ibvp), say with the unknown $\hat{u}$, such that this new ibvp (observer system) uses only the partial information extracted from the original system written with the unknown u, and $u - \hat{u}$ tends to zero in a physically* *meaningful sense? Moreover, can you construct a controller for the observer system which uses only the state $\hat{u}$ such that the same controller also stabilizes the original plant?*

Remark 1.4. *It is important to notice that if there is no access to the medium, we may not even be able to measure the initial profile u_0 of the original plant. All we may know is how regular this initial profile is and some partial information about it, such as some of its boundary traces or some of its values at some points within the domain. Therefore, an answer to Problem 1.3 serves two purposes: (i) We (approximately) reconstruct the solution of the original plant; (ii) we find a feedback stabilizer which uses the full state of a different system that we can measure completely.*

1.1. **A few words on the literature.** Recently, (Marx & Cerpa, 2018) constructed an observer for the Korteweg-de Vries equation subject to the boundary conditions $u(0,t) = U(t)$, $u_x(L,t) = u_{xx}(L,t) = 0$ by using the partial measurement $y(t) = u(L,t)$. Here the left end boundary input $U(t)$ is a controller (stabilizer) obtained by using the backstepping method. This controller uses only the state values of the observer, say $\hat{u}$. Prior to this work, the same authors (Marx & Cerpa, 2014) constructed an observer for the linearized Korteweg-de Vries equation subject to the boundary conditions $u(0,t) = U(t)$, $u(L,t) = u_x(L,t) = 0$ by using the partial measurement $y(t) = u_{xx}(L,t)$. The observer design for the corresponding nonlinear model was studied by (Hasan, 2016). As we have indicated, the backstepping controllability problem with the same set of boundary conditions gets more challenging if one tries to control the system from the right endpoint. One remedy is to put only one boundary condition at the right endpoint where the control acts to, and consider two boundary conditions at the left endpoint. Backstepping stabilization for the linearized KdV equation with such a set of boundary conditions was studied by (Tang & Krstic, 2013a) and (Tang & Krstic, 2013b). However, usually the boundary conditions are determined by the intrinsic nature of the physical model, and one may not be able to choose the number of boundary conditions at a particular endpoint. The novelty of the present article is that we are able to construct an observer and the corresponding boundary feedback stabilizers acting from the opposite of the endpoint where only one boundary condition is specified.

We should also mention some important work related to the control and stabilization of the KdV equation. Exact boundary controllability of the linear and nonlinear KdV equations with the same type of boundary conditions as in (1.1) was studied by (Rosier, 1997); (Coron & Crépeau, 2004), (Zhang, 1999), (Glass & Guerrero, 2008), (Cerpa, 2007), (Cerpa & Crépeau, 2009), (Rosier & Zhang, 2009), and (Glass & Guerrero, 2010). Stabilization of solutions of the KdV equation with a localised interior damping was achieved by (Perla Menzala et al., 2002), (Pazoto, 2005), and (Massarolo et al., 2007). (Balogh & Krstic, 2000). There are also some results achieving stabilization of the KdV equation by using predetermined local boundary feedbacks, see for instance (W.-J. Liu & Krstić, 2002) and (Jia, 2016).

1.2. **Preliminaries and main result.** Before we state our main results, let us give some important facts and notations that will be needed later.

Let η be a C^∞-function and $\Upsilon_\eta : H^l(\Omega) \to H^l(\Omega)$ $(l \geq 0)$ be the integral operator defined by

$$(\Upsilon_\eta \varphi)(x) := \int_0^x \eta(x,y)\varphi(y)dy,$$

where $H^l(\Omega)$ denotes the L^2-based Sobolev spaces. Then the following result holds true (W. Liu, 2003),(Özsarı & Batal, 2018):

Lemma 1.5. *$I - \Upsilon_\eta$ is invertible with a bounded inverse from $H^l(\Omega) \to H^l(\Omega)$ $(l \geq 0)$. Moreover, $(I - \Upsilon_\eta)^{-1}$ can be written as $I + \Phi$, where Φ is a bounded operator from $L^2(\Omega)$ into $H^l(\Omega)$ for $l = 0, 1, 2$ and from $H^{l-2}(\Omega)$ into $H^l(\Omega)$ for $l > 2$.*

We define the following (error) state space, which will be useful in a moment.

$$E := \left\{\varphi \in H^3(\Omega) \,|\, \varphi(0) = \varphi(L) = \varphi'(L) = 0\right\}. \tag{1.6}$$

We also introduce the following notation for representing solution spaces for $s \geq 0$:

$$X_T^s = C([0,T]; H^s(\Omega)) \cap L^2(0,T; H^{s+1}(\Omega)).$$

Theorem 1.7. *Let $T > 0$, $u_0, \hat{u}_0 \in L^2(\Omega)$ such that $u_0 - \hat{u}_0 \in E$, and let p and k be the smooth kernels solving* (2.5) *and* (2.25)*, respectively. Then, the plant-observer-error (POE) system given in* (1.1)*,* (2.1)*,* (2.2) *has a solution $(u, \hat{u}, \tilde{u}) \in X_T^0 \times X_T^0 \times X_T^3$ with right endpoint boundary controllers*

$$U(t) := [\Upsilon_k \hat{u}](L,t) \text{ and } V(t) := [\Upsilon_{k_x} \hat{u}](L,t).$$

Moreover, there exist $\alpha > \kappa > 0$ such that the decay rate estimates

$$\|\hat{u}\|_{L^2(\Omega)} \lesssim \left(\|\hat{u}_0\|_{L^2(\Omega)} + \|u_0 - \hat{u}_0\|_{H^3(\Omega)}\right) e^{-\kappa t}, \tag{1.8}$$

$$\|u - \hat{u}\|_{L^2(\Omega)} \lesssim \|u_0 - \hat{u}_0\|_{L^2(\Omega)} e^{-\alpha t}, \tag{1.9}$$

$$\|u - \hat{u}\|_{H^3(\Omega)} \lesssim \|u_0 - \hat{u}_0\|_{H^3(\Omega)} e^{-\alpha t} \tag{1.10}$$

hold true for $t \in [0,T]$.

2. Linearized model

2.1. **Observer design.** We introduce the following *observer* system:

$$\begin{cases} \hat{u}_t + \hat{u}_x + \hat{u}_{xxx} & \\ +P_1(x)\left(u_x(0,t) - \hat{u}_x(0,t)\right) + P_2(x)\left(u_{xx}(0,t) - \hat{u}_{xx}(0,t)\right) = 0 & \text{in } \Omega \times (0,T), \\ \hat{u}(0,t) = 0,\ \hat{u}(L,t) = \hat{U}(t),\ \hat{u}_x(L,t) = \hat{V}(t) & \text{in } (0,T), \\ \hat{u}(x,0) = \hat{u}_0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here $P_1(x)$ and $P_2(x)$ are observer gains to be determined. Our objective is to construct an efficient observer by finding the observer gains $P_1(x)$ and $P_2(x)$ so that u converges to $\hat{u}$ as t gets larger. This is equivalent to showing that the error function, which is defined by $\tilde{u} \equiv u - \hat{u}$ goes to zero as t gets larger. Note that the error satisfies the PDE model given by

$$\begin{cases} \tilde{u}_t + \tilde{u}_x + \tilde{u}_{xxx} = P_1(x)\tilde{u}_x(0,t) + P_2(x)\tilde{u}_{xx}(0,t) \text{ in } \Omega \times (0,T); \\ \tilde{u}(0,t) = 0, \tilde{u}(L,t) = 0, \tilde{u}_x(L,t) = 0 \text{ in } (0,T); \\ \tilde{u}(x,0) = u_0(x) - \hat{u}_0(x) \text{ in } \Omega. \end{cases} \tag{2.2}$$

We want to find a transformation in the form

$$\tilde{u}(x,t) = \tilde{w}(x,t) - \int_0^x p(x,y)\tilde{w}(y,t)dy \tag{2.3}$$

which maps the solution of the equation

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \tilde{\lambda}\tilde{w} = 0, \text{ in } \Omega \times (0,T), \\ \tilde{w}(0,t) = 0, \ \tilde{w}(L,t) = 0, \ \tilde{w}_x(L,t) = \int_0^L p_x(L,y)\tilde{w}(y,t)dy, \text{ in } (0,T), \\ \tilde{w}(x,0) = \tilde{w}_0(x), \text{ in } \Omega \end{cases} \tag{2.4}$$

to the error system (2.2) for some $\tilde{\lambda} > 0$. Computing the relevant partial derivatives of both sides of (2.3), applying integration by parts and using the given boundary conditions it can be shown that the desired transformation (2.3) is obtained if $p(x,y)$ satisfies the PDE

$$\begin{cases} p_{xxx} + p_{yyy} + p_x + p_y = \tilde{\lambda}p, \quad x, y \in \Omega, \\ p(L,y) = 0, \ p(x,x) = 0, \\ p_x(x,x) = -\frac{\tilde{\lambda}}{3}(x-L), \end{cases} \tag{2.5}$$

with the choice of

$$P_1(x) = p_y(x,0), \ P_2(x) = -p(x,0),$$

$$\tilde{u}(x,0) = u_0(x) - \hat{u}_0(x) = -\int_0^L p(x,y)\tilde{w}_0(y)dy.$$

Note that with the change of variables $\tilde{x} \equiv L - y$ and $\tilde{y} \equiv L - x$ and $k(\tilde{x},\tilde{y}) = p(x,y)$, it is easy to see that k is the C^∞ pseudo kernel (Özsarı & Batal, 2018, Lemma 2.2) which solves (2.25), where x, y, and λ, replaced by $\tilde{x}$, $\tilde{y}$, and $\tilde{\lambda}$.

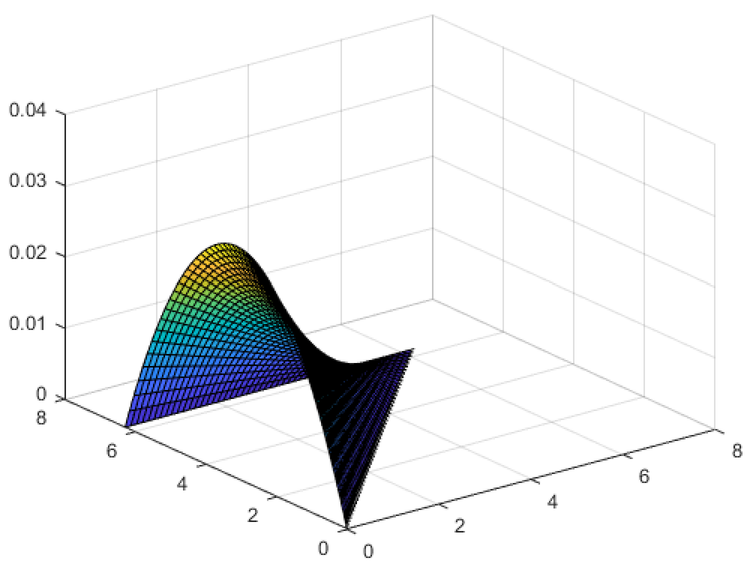

FIGURE 1. Pseudo-kernel p on a domain of length 2π and $\lambda = 0.1$.

Note also that by this transformation we see that $p_x(L,y) = -k_{\tilde{y}}(\tilde{x},0)$, and in (Özsarı & Batal, 2018, Lemma2.36) it is shown that for suitably small, $\tilde{\lambda} > 0$, the quantity

$$\tilde{\lambda} - \frac{1}{2}\|k_{\tilde{y}}(\cdot,0)\|^2_{L^2(\Omega)}$$

is strictly greater than zero. Therefore choosing $\tilde{\lambda}$ sufficiently small, we can guarantee that $\alpha \equiv \tilde{\lambda} - \frac{1}{2}\|p_x(L,\cdot)\|^2_{L^2(\Omega)} > 0$. We need the following lemma:

Lemma 2.6. *Let $\tilde{w}$ be the solution of* (2.4)*. Then the following inequalities hold:*

$$\|\tilde{w}\|_{L^2(\Omega)} \le \|\tilde{w}_0\|_{L^2(\Omega)} e^{-\alpha t}, \tag{2.7}$$

$$|\tilde{w}_x(0,t)| + |\tilde{w}_{xx}(0,t)| + \|\tilde{w}\|_{H^3(\Omega)} \lesssim \|\tilde{w}_0\|_{H^3(\Omega)} e^{-\alpha t}. \tag{2.8}$$

Proof. We multiply (2.4) by $\tilde{w}$ and integrate over Ω. Applying integration by parts and boundary conditions we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}(t)\|^2_{L^2(\Omega)} + \tilde{\lambda}\|\tilde{w}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}|\tilde{w}_x(0,t)|^2 = \frac{1}{2}|\tilde{w}_x(L,t)|^2, \tag{2.9}$$

which, together with (2.4), implies

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}(t)\|^2_{L^2(\Omega)} + \tilde{\lambda}\|\tilde{w}(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\left(\int_0^L p_x(L,y)\tilde{w}(y,t)dy\right)^2. \tag{2.10}$$

Applying the Cauchy-Schwarz inequality to the right hand side we see that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}(t)\|^2_{L^2(\Omega)} + \big(\tilde{\lambda} - \frac{1}{2}\|p_x(L,\cdot)\|^2_{L^2(\Omega)}\big)\|\tilde{w}(t)\|^2_{L^2(\Omega)} \le 0, \tag{2.11}$$

which implies (2.7).

In order to prove (2.8), we first differentiate (2.4) with respect to t, then multiply by $\tilde{w}_t$ and integrate over Ω. Using integration by parts and boundary conditions as well, we see that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} + \tilde{\lambda}\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2}|\tilde{w}_{tx}(0,t)|^2 = \frac{1}{2}|\tilde{w}_{tx}(L,t)|^2. \tag{2.12}$$

Moreover by (2.4) we have $\tilde{w}_{tx}(L,t) = \int_0^L p_x(L,y)\tilde{w}_t(y,t)dy$. Hence we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} + \tilde{\lambda}\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\left(\int_0^L p_x(L,y)\tilde{w}_t(y,t)dy\right)^2. \tag{2.13}$$

Applying the Cauchy-Schwarz inequality to the right hand side we see that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} + \big(\tilde{\lambda} - \frac{1}{2}\|p_x(L,\cdot)\|^2_{L^2(\Omega)}\big)\|\tilde{w}_t(t)\|^2_{L^2(\Omega)} \le 0, \tag{2.14}$$

which implies

$$\|\tilde{w}_t(t)\|_{L^2(\Omega)} \le \|\tilde{w}_t(0)\|_{L^2(\Omega)} e^{-\alpha t} \le \|\tilde{w}_0\|_{H^3(\Omega)} e^{-\alpha t} \tag{2.15}$$

since $\|\tilde{w}_t(0)\|_{L^2(\Omega)} = \|\tilde{w}_0' + \tilde{w}_0''' + \tilde{\lambda}\tilde{w}_0\| \le \|\tilde{w}_0\|_{H^3(\Omega)}$. On the other hand, by (2.4) we also have

$$\|\tilde{w}_{xxx}(t)\|^2_{L^2(\Omega)} \le 3\big(\|\tilde{w}_x(t)\|^2_{L^2(\Omega)} + \tilde{\lambda}\|\tilde{w}(t)\|^2_{L^2(\Omega)} + \|\tilde{w}_t(t)\|^2_{L^2(\Omega)}\big). \tag{2.16}$$

Applying ϵ-Young's inequality to the square of the right hand side of the Gagliardo-Nirenberg inequality

$$\|\tilde{w}_x(t)\|_{L^2(\Omega)} \le \|\tilde{w}_{xxx}(t)\|^{\frac{1}{3}}_{L^2(\Omega)}\|\tilde{w}(t)\|^{\frac{2}{3}}_{L^2(\Omega)}, \tag{2.17}$$

we also obtain

$$\|\tilde{w}_x(t)\|^2_{L^2(\Omega)} \le \epsilon\|\tilde{w}_{xxx}(t)\|^2_{L^2(\Omega)} + c_\epsilon\|\tilde{w}(t)\|^2_{L^2(\Omega)} \tag{2.18}$$

for $\epsilon > 0$. Combining (2.16) and (2.18), and choosing ϵ small enough, we see that

$$\|\tilde{w}_{xxx}(t)\|_{L^2(\Omega)} \lesssim \|\tilde{w}(t)\|_{L^2(\Omega)} + \|\tilde{w}_t(t)\|_{L^2(\Omega)}. \tag{2.19}$$

Hence

$$\|\tilde{w}(t)\|_{H^3(\Omega)} \lesssim \|\tilde{w}(t)\|_{L^2(\Omega)} + \|\tilde{w}_t(t)\|_{L^2(\Omega)}, \tag{2.20}$$

which, together with (2.7) and (2.15), implies

$$\|\tilde{w}\|_{H^3(\Omega)} \lesssim \|\tilde{w}_0\|_{H^3(\Omega)} e^{-\alpha t}. \tag{2.21}$$

To obtain the second part of inequality (2.8) we multiply (2.4) by $(L-x)\tilde{w}_{xx}$ and integrate over Ω. Applying integration by parts and boundary conditions we obtain

$$\tilde{w}_x^2(0,t) + \tilde{w}_{xx}^2(0,t) = \frac{2}{L}\int_0^L \Big((L-x)\tilde{w}_t\tilde{w}_{xx} + \frac{1}{2}\tilde{w}_x^2 + \frac{1}{2}\tilde{w}_{xx}^2 + \tilde{\lambda}(L-x)\tilde{w}\tilde{w}_{xx}\Big)dx. \tag{2.22}$$

Using Cauchy-Schwarz and Young's inequalities on the first and last term of the right hand side we see that

$$|\tilde{w}_x(0,t)|^2 + |\tilde{w}_{xx}(0,t)|^2 \lesssim \|\tilde{w}_t\|^2_{L^2(\Omega)} + \|\tilde{w}\|^2_{H^3(\Omega)}, \tag{2.23}$$

which, together with (2.15) and (2.21), implies (2.8). □

Now for $\hat{u}$, let us apply the pseudo-backstepping transformation

$$\hat{w} = \hat{u} - \int_0^x k(x,y)\hat{u}(y,t)dy \tag{2.24}$$

where $k(x,y)$ is a C^∞ pseudo kernel (Özsarı & Batal, 2018, Lemma 2.2) which solves

$$\begin{aligned} k_{xxx} + k_{yyy} + k_y + k_x &= -\lambda k, \\ k(x,x) = k(x,0) &= 0, \\ k_x(x,x) &= \frac{\lambda}{3}x, \end{aligned} \tag{2.25}$$

where the PDE model is considered on the triangular spatial domain

$$\mathcal{T} \equiv \{(x,y) \in \mathbb{R}^2 \,|\, x \in [0,L], y \in [0,x]\}.$$

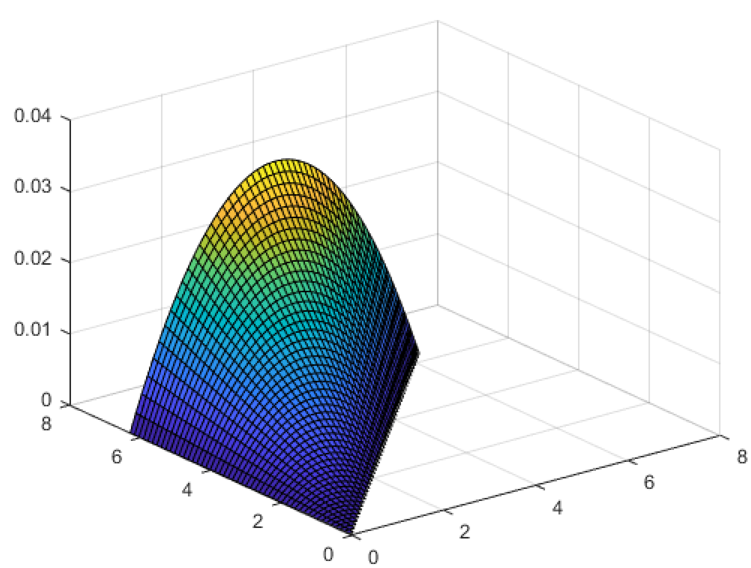

FIGURE 2. Pseudo-kernel k on a domain of length 2π and $\lambda = 0.1$.

Then choosing

$$\begin{aligned} U(t) &= \textstyle\int_0^L k(L,y)\hat{u}(y,t)dy, \\ V(t) &= \textstyle\int_0^L k_x(L,y)\hat{u}(y,t)dy. \end{aligned} \tag{2.26}$$

we obtain the following equation for $\hat{w}$:

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \lambda\hat{w} \\ = k_y(x,0)\hat{w}_x(0,t) - \Psi_1(x)\tilde{w}_x(0,t) - \Psi_2(x)\tilde{w}_{xx}(0,t), \\ \hat{w}(0,t) = 0,\ \hat{w}(L,t) = 0,\ \hat{w}_x(L,t) = 0, \end{cases} \tag{2.27}$$

where

$$\Psi_i(x) \equiv P_i(x) - \int_0^x P_i(y)k(x,y)dy,$$

for $i \in \{1,2\}$. Multiplying (2.27) by $\hat{w}$ and integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{w}(t)\|^2_{L^2(\Omega)} + \lambda\|\hat{w}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}|\hat{w}_x(0,t)|^2 \\ = \hat{w}_x(0,t)\int_0^L k_y(x,0)\hat{w}(x,t)dx \\ - \tilde{w}_x(0,t)\int_0^L \Psi_1(x)\hat{w}(x,t)dx - \tilde{w}_{xx}(0,t)\int_0^L \Psi_2(x)\hat{w}(x,t)dx. \end{aligned} \tag{2.28}$$

Applying ϵ-Young's and Cauchy-Schwarz inequalities to the right hand side, for any ϵ we get

$$\frac{1}{2}\frac{d}{dt}\|\hat{w}(t)\|^2_{L^2(\Omega)} + \kappa\|\hat{w}(t)\|^2_{L^2(\Omega)} \leq \frac{1}{2\epsilon}\left[\tilde{w}_x^2(0,t) + \tilde{w}_{xx}^2(0,t)\right], \tag{2.29}$$

where

$$\kappa \equiv \lambda - \frac{1}{2}\|k_y(\cdot,0)\|^2_{L^2(\Omega)} - \frac{1}{2}\epsilon\big(\|\Psi_1\|^2_{L^2(\Omega)} + \|\Psi_2\|^2_{L^2(\Omega)}\big).$$

By (Özsarı & Batal, 2018, Lemma2.36) we know that for sufficiently small λ, the quantity $\lambda - \frac{1}{2}\|k_y(\cdot,0)\|^2_{L^2(\Omega)} > 0$. Therefore choosing ϵ sufficiently small we can make the coefficient $\kappa > 0$. Moreover since $\alpha \equiv \tilde{\lambda} - \frac{1}{2}\|p_x(L,\cdot)\|^2_{L^2(\Omega)}$ and $p_x(L,y) = k_y(x,0)$ choosing $\lambda = \tilde{\lambda}$ if necessary, we can assume $\alpha > \kappa$. Inequality (2.29) and (2.8) imply

$$\frac{1}{2}\frac{d}{dt}\|\hat{w}(t)\|^2_{L^2(\Omega)} + \kappa\|\hat{w}(t)\|^2_{L^2(\Omega)} \lesssim \|\tilde{w}_0\|^2_{H^3(\Omega)}e^{-2\alpha t}. \tag{2.30}$$

Using $\alpha > \kappa$, multiplying both sides of above inequality by $e^{2\kappa t}$ and taking integral of both sides from 0 to t we can easily see that

$$\|\hat{w}(t)\|^2_{L^2(\Omega)} \lesssim \left(\|\hat{w}_0\|^2_{L^2(\Omega)} + \|\tilde{w}_0\|^2_{H^3(\Omega)}\right)e^{-2\kappa t},$$

which is equivalent to saying

$$\|\hat{w}(t)\|_{L^2(\Omega)} \lesssim \left(\|\hat{w}_0\|_{L^2(\Omega)} + \|\tilde{w}_0\|_{H^3(\Omega)}\right)e^{-\kappa t}. \tag{2.31}$$

On the other hand both of the transformations given in (2.3) and (2.24) are bounded with bounded inverses (W. Liu, 2003, Lemma 2.4), (Özsarı & Batal, 2018, Lemma 2.22). Therefore, we have

$$\begin{aligned} \|\tilde{u}(t)\|_{H^3(\Omega)} \lesssim \|\tilde{w}(t)\|_{H^3(\Omega)}, \quad \|\tilde{w}_0\|_{H^3(\Omega)} \lesssim \|\tilde{u}_0\|_{H^3(\Omega)}, \\ \|\hat{u}(t)\|_{L^2(\Omega)} \lesssim \|\hat{w}(t)\|_{L^2(\Omega)}, \quad \|\hat{w}_0\|_{L^2(\Omega)} \lesssim \|\hat{u}_0\|_{L^2(\Omega)}. \end{aligned} \tag{2.32}$$

Combining (2.31) and (2.32) we achieve

$$\|\hat{u}\|_{L^2(\Omega)} \lesssim \left(\|\hat{u}_0\|_{L^2(\Omega)} + \|u_0 - \hat{u}_0\|_{H^3(\Omega)}\right) e^{-\kappa t}. \tag{2.33}$$

Moreover (2.7), (2.8) and (2.32) also imply

$$\|u - \hat{u}\|_{L^2(\Omega)} \lesssim \|u_0 - \hat{u}_0\|_{L^2(\Omega)} e^{-\alpha t}, \tag{2.34}$$

$$\|u - \hat{u}\|_{H^3(\Omega)} \lesssim \|u_0 - \hat{u}_0\|_{H^3(\Omega)} e^{-\alpha t}. \tag{2.35}$$

Hence u itself decays exponentially.

2.2. **Wellposedness.** We will prove the wellposedness assuming $L = 1$ to simplify the discussion. The general case can be treated in a very similar way. We will first prove the wellposedness of the target error system (2.4). Let us start by considering the following problem:

$$\begin{cases} q_t + q_x + q_{xxx} + \tilde{\lambda} q = 0, \text{ in } \Omega \times (0,T), \\ q(0,t) = 0, \, q(L,t) = 0, \, q_x(L,t) = h(t), \text{ in } (0,T), \\ q(x,0) = q_0(x), \text{ in } \Omega, \end{cases} \tag{2.36}$$

where $h \in H^1(0,T)$, $q_0 \in H^3(\Omega)$ satisfy the compatibility conditions $q_0(0) = 0, q_0(L) = 0$, and $q_0'(L) = h(0)$. The well-posedness of (2.36) can be obtained as in (Bona, Sun, & Zhang, 2003, Lemma 3.3) and one has $q \in X_T^3$ together with $q_t \in X_T^0$. Note that in (2.4), the boundary condition $\tilde{w}_x(L,t) = \int_0^L p_x(L,y)\tilde{w}(y,t)dy$ is of feedback type. We will treat this by using a fixed point argument. To this end, we consider the following feedback problem instead of (2.36):

$$\begin{cases} q_t + q_x + q_{xxx} + \tilde{\lambda} q = 0, \text{ in } \Omega \times (0,T), \\ q(0,t) = 0, \, q(L,t) = 0, \, q_x(L,t) = h(q)(t), \text{ in } (0,T), \\ q(x,0) = q_0(x), \text{ in } \Omega, \end{cases} \tag{2.37}$$

where $h(q)(t) = \int_0^L p_x(L,y)q(y,t)dy$, $q_0 \in H^3(\Omega)$ with the compatibility conditions $q_0(0) = 0, q_0(L) = 0$, and $q_0'(L) = h(q)(0) = \int_0^L p_x(L,y)q_0(y)dy$. Let us define the Banach space $Q_T \equiv \{q \in X_T^3 \,|\, q_t \in X_T^0\}$ and the complete metric subspace of it $\tilde{Q}_T = \{q \in Q_T \,|\, q(\cdot,0) = q_0(\cdot)\}$ with the metric induced from the norm of Q_T. Observe that given $q^* \in \tilde{Q}_T$, since p is a smooth solution of (2.5), one has $h(q^*)(\cdot) = \int_0^L p_x(L,y)q^*(y,\cdot)dy \in H^1(0,T)$. Therefore, by the well-posedness of the linear system, this implicitly defines an operator $\Gamma : \tilde{Q}_T \to \tilde{Q}_T$ given by $\Gamma(q^*) = q$. Now we will show that Γ has a fixed point. Let $q_1, q_2 \in \tilde{Q}_T$. Using the linear estimates given in (Bona et al., 2003, Lemma 3.3), we have

$$\begin{aligned} d(\Gamma(q_1), \Gamma(q_2))_{\tilde{Q}_T} &= \|\Gamma(q_1) - \Gamma(q_2)\|_{Q_T} \\ &\leq C\|h(q_1)(\cdot) - h(q_2)(\cdot)\|_{H^1(0,T)} \leq CT\|q_1 - q_2\|_{Q_T} = CTd(q_1,q_2)_{\tilde{Q}_T}. \end{aligned} \tag{2.38}$$

By using the Banach fixed point theorem, we obtain the existence of a unique solution $q \in Q_T$ for small T. This implies the local well-posedness for (2.4). The fact that this local solution is indeed global follows from the uniform bounds obtained in the proof of stabilization. Now, by using the transformation in (2.3), we obtain the wellposedness of the error system (2.2). We know by Lemma 2.6 above that $\tilde{w}_x(0,\cdot), \tilde{w}_{xx}(0,\cdot) \in L^2(0,T)$.

Therefore, the right hand side of (2.27) can be written as $a(x)\hat{w}_x(0,t) + f(x,t)$ with $a(x) = k_y(x,0)$ and $f(x,t) = -\Psi_1(x)\tilde{w}_x(0,t) - \Psi_2(x)\tilde{w}_{xx}(0,t)$ such that $f \in L^2(0,T;H^\infty(\Omega))$. Well-posedness of this problem was studied in (Özsarı & Batal, 2018, Remark 3.23) and for given $\hat{w}_0 \in L^2(\Omega)$, one has $\hat{w} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$. Now, by the invertibility of (2.24), we obtain the wellposedness of the observer system (2.1) so that

$$\hat{u} \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)). \tag{2.39}$$

Combining the wellposedness of (2.2) and (2.1), we obtain the wellposedness of the original system and conclude that $u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$.

3. Numerics

3.1. **Algorithm.** In this section, we describe the steps to obtain the numerical solution of the plant-observer-error system given in (1.1), (2.1), and (2.2). We follow a different approach compared to for instance (Marx & Cerpa, 2018). Our idea is based on first solving the models (2.2) and (2.27) with homogeneous boundary conditions and then obtaining the solutions of nonhomogeneous boundary value problems (1.1) and (2.1) by using the invertibility of the backstepping transformation given in Lemma 1.5.

(Step 1): At first we obtain numerical solutions of kernel models (2.25) and (2.5). This is done via successive approximation. More precisely, we first change variables by setting $t \equiv y$, $s \equiv x - y$, and $G(s,t) \equiv k(x,y)$. Then, G satisfies the boundary value problem given by

$$\begin{aligned} G_{ttt} - 3G_{stt} + 3G_{sst} + G_t &= -\lambda G, && (3.1)\\ G(s,0) = G(0,t) &= 0, && (3.2)\\ G_s(0,t) &= \frac{\lambda}{3} t && (3.3) \end{aligned}$$

on the triangular domain $\mathcal{T}_0 \equiv \{(s,t) \,|\, t \in [0,L], s \in [0, L-t]\}$. Note that the solution of (3.1)-(3.3) can be constructed by solving the integral equation

$$G(s,t) = \frac{\lambda}{3} st + \frac{1}{3} \int_0^t \int_0^s \int_0^\omega (-G_{ttt} + 3G_{stt} - G_t - \lambda G)(\xi,\eta) d\xi d\omega d\eta. \tag{3.4}$$

Therefore, we set

$$G^n(s,t) = \frac{\lambda}{3} st + \frac{1}{3} \int_0^t \int_0^s \int_0^\omega (-G^{n-1}_{ttt} + 3G^{n-1}_{stt} - G^{n-1}_t - \lambda G^{n-1})(\xi,\eta) d\xi d\omega d\eta$$

for $n \geq 1$ with $G^0 \equiv 0$. We have proven in (Özsarı & Batal, 2018) that the sequence G^n uniformly converges to a smooth function on $\mathcal{T}_0$. For the sake of numerical experiments, we define a parameter $n_{iter} \in \mathbb{Z}_+$ and use

$$k_{num}(x,y) = G^{n_{iter}}(x-y, y)$$

for the kernel k. Since the solution of (2.5) is given by $p(x,y) = k(L-y, L-x)$, we will use

$$p_{num}(x,y) = k_{num}(L-y, L-x) = G^{n_{iter}}(x-y, L-x)$$

for the kernel p. The observer gains P_1 and P_2 will then be taken as

$$P_{1,num}(x) = \frac{\partial}{\partial_y} p_{num}(x,0) \text{ and } P_{2,num}(x,0) = -p_{num}(x,0).$$

Using these polynomial approximations, we also define approximations for Ψ_i, $i = 1, 2$, by setting

$$\Psi_{i,num}(x) \equiv P_{i,num}(x) - \int_0^x P_{i,num}(y)k_{num}(x,y)dy.$$

(Step 2): Secondly, we numerically solve the error system (2.2). In order to do this, we modify the finite difference scheme given in (Pazoto et al., 2010). To this end, we set the discrete space

$$X_J := \{\tilde{u} = (\tilde{u}_0, \tilde{u}_1, ..., \tilde{u}_J) \in \mathbb{R}^{J+1} \,|\, \tilde{u}_0 = \tilde{u}_{J-1} = \tilde{u}_J = 0\},$$

and the difference operators $(D^+\tilde{u})_j := \dfrac{\tilde{u}_{j+1} - \tilde{u}_j}{\delta x}$, $(D^-\tilde{u})_j := \dfrac{\tilde{u}_j - \tilde{u}_{j-1}}{\delta x}$ for $j = 1, ..., J-1$, and $D = \dfrac{1}{2}(D^+ + D^-)$. Let δx and δt be the space and time steps for $j = 0, ..., J$, and $n = 0, 1, ..., N$, respectively. Then the numerical approximation of the linearised error system (2.2) takes the form

$$\frac{\tilde{u}_j^{n+1} - \tilde{u}_j^n}{\delta t} + (\mathcal{A}\tilde{u}^{n+1})_j = P_{1,num}(x_j)\frac{\tilde{u}_1^n}{\delta x} + P_{2,num}(x_j)\frac{(\tilde{u}_2^n - 2\tilde{u}_1^n)}{(\delta x)^2}, \quad j = 1, ..., J-1 \quad (3.5)$$

$$\tilde{u}_0 = \tilde{u}_{J-1} = \tilde{u}_J = 0, \quad (3.6)$$

$$\tilde{u}_0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{u}_0(x)dx, \quad j = 1, ..., J-1, \quad (3.7)$$

where $x_{j\mp\frac{1}{2}} = (j \mp \frac{1}{2})\delta x$, $x_j = j\delta x$. The $(J-1) \times (J-1)$ matrix $\mathcal{A}$ approximates $\tilde{u}_x + \tilde{u}_{xxx}$ and it is defined by $\mathcal{A} := D^+D^+D^- + D$. Let us set $\tilde{\mathcal{C}} := I + \delta t A$. Then, from the main equation, we obtain

$$\tilde{u}_j^{n+1} = \tilde{\mathcal{C}}^{-1}\left(\tilde{u}_j^n + P_{1,num}(x_j)\frac{(\delta t)\tilde{u}_1^n}{\delta x} + P_{2,num}(x_j)\frac{\delta t(\tilde{u}_2^n - 2\tilde{u}_1^n)}{(\delta x)^2}\right)$$

for $j = 1, ..., J-1$.

(Step 3): The next step is to solve (2.27). The right hand side of the main equation in (2.27) includes the traces $\tilde{w}_x(0,t)$ and $\tilde{w}_{xx}(0,t)$. Observe that these traces are equal to $\tilde{u}_x(0,t)$ and $\tilde{u}_{xx}(0,t)$ by the transformation (2.3) and the boundary conditions $p(x,x) = 0$ and $\tilde{w}(0,t) = 0$. Therefore, we can use the approximations $\dfrac{\tilde{u}_1^n}{\delta x}$ and $\dfrac{\tilde{u}_2^n - 2\tilde{u}_1^n}{(\delta x)^2}$ from the previous step to approximate $\tilde{u}_x(0,t_n)$ and $\tilde{u}_{xx}(0,t_n)$ at the n^{th} time step. Then the numerical approximation of the linearised observer target system (2.27) takes the form

$$\frac{\hat{w}_j^{n+1} - \hat{w}_j^n}{\delta t} + (\mathcal{A}\hat{w}^{n+1})_j + \lambda\hat{w}^{n+1} = (RHS), \quad j = 1, ..., J-1 \quad (3.8)$$

$$\hat{w}_0 = \hat{w}_{J-1} = \hat{w}_J = 0, \quad (3.9)$$

$$\hat{w}_0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \hat{w}_0(x)dx, \quad j = 1, ..., J-1, \quad (3.10)$$

where $\hat{w}_0$ is obtained from the transformation (2.24) and

$$(RHS) = \frac{\partial}{\partial_y} k_{num}(x_j, 0) - \Psi_{1,num}(x_j)\frac{\tilde{u}_1^n}{\delta x} - \Psi_{2,num}(x_j)\frac{(\tilde{u}_2^n - 2\tilde{u}_1^n)}{(\delta x)^2}.$$

Let us set $\hat{\mathcal{C}} := (1 + \delta t\lambda)I + \delta t A$. Then, from the main equation, we obtain

$$\hat{w}_j^{n+1} = \hat{\mathcal{C}}^{-1}\left(\hat{w}_j^n + \frac{\partial}{\partial_y} k_{num}(x_j, 0) - \Psi_{1,num}(x_j)\frac{\tilde{u}_1^n}{\delta x} - \Psi_{2,num}(x_j)\frac{(\tilde{u}_2^n - 2\tilde{u}_1^n)}{(\delta x)^2}\right)$$

for $j = 1, ..., J - 1$.

In order to obtain the solution of the observer system (2.1), we use the inverse of the transformation (2.24). Given $\tilde{w}$, we can find the corresponding inverse image $\hat{u}$ via the succession method given in the proof of Lemma 1.5 (see for example (W. Liu, 2003, Lemma 2.4) and (Özsarı & Batal, 2018, Lemma 2.22)). To this end, let m_{iter} denote the number of iterations in the succession and set $v^0 = \mathcal{K}\tilde{w}$, $v^k := \mathcal{K}(\tilde{w} + v^{k-1})$ for $1 \le k \le m_{iter}$, where $\mathcal{K}$ is the numerical approximation of the integral in the definition of K. Then, $v^{m_{iter}}$ is an approximation of $v = \Phi(\tilde{w})$, and one gets an approximation of the solution of the observer system by setting $\hat{u}(x_j, t_n) := \hat{w}(x_j, t_n) + v^{m_{iter}}(x_j, t_n)$.

(Step 4): Finally, we solve the original plant (1.1) by setting

$$u(x_j, t_n) := \hat{u}(x_j, t_n) + \tilde{u}(x_j, t_n).$$

3.2. **Simulations.** In this section, we give two simulations for the linear model: (i) uncontrolled solution (ii) controlled solution.

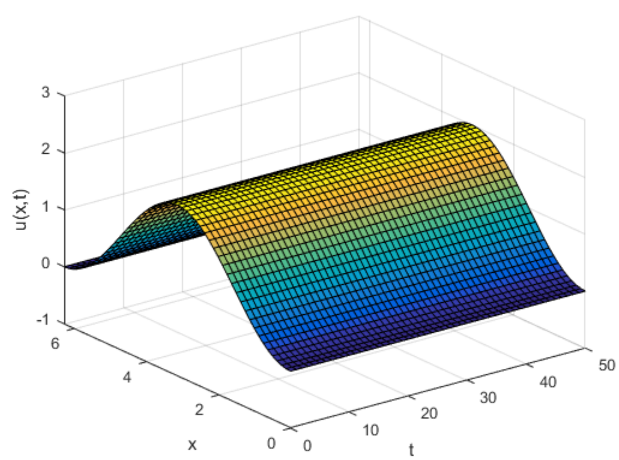

FIGURE 3. Uncontrolled solution with initial datum $u_0 = 1 - \cos(x)$ on a domain of length 2π.

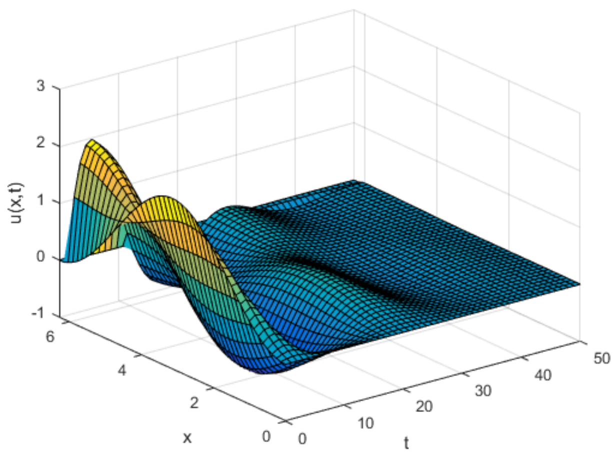

FIGURE 4. Controlled solution with initial data $u_0 = 1-\cos(x)$, $\hat{u}_0 = 0$, kernel parameters $\lambda = \tilde{\lambda} = 0.01$ on a domain of length $L = 2\pi$, $n_{iter} = m_{iter} = 10$.

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