

Pseudo-backstepping and its application to the control of Korteweg-de Vries equation from the right endpoint on a finite domain

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ABSTRACT. In this paper, we design Dirichlet-Neumann boundary feedback controllers for the Korteweg-de Vries (KdV) equation which act at the right endpoint of the domain. Controlling the KdV equation from the right endpoint of the domain is a mathematically more challenging problem than its left endpoint counterpart from the point of constructing backstepping controllers. The standard application of the backstepping method fails because corresponding kernel models become overdetermined. In order to deal with this difficulty we introduce the *pseudo-backstepping* method which uses a *pseudo-kernel* that satisfies all but one desirable boundary condition. Moreover, various norms of the pseudo-kernel can be controlled through a parameter in one of its boundary conditions. We are able to prove that the boundary controllers constructed via this pseudo-kernel still exponentially stabilize the system with the cost of a low exponential rate of decay. At the end of the paper, we give numerical simulations to illustrate our main result.

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1. INTRODUCTION

This article is devoted to the study of the boundary feedback stabilization of the Korteweg-de Vries (KdV) equation on a bounded domain $\Omega = (0, L) \subset \mathbb{R}$. The linear version of the model under consideration is given by

$$(1.1) \quad \begin{cases} u_t + u_x + u_{xxx} = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u(0, t) = 0, u(L, t) = U(t), u_x(L, t) = V(t) & \text{in } \mathbb{R}_+, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

whereas the nonlinear version of this model is written with the main equation in (1.1) replaced with

$$(1.2) \quad u_t + u_x + u_{xxx} + uu_x = 0.$$

In (1.2), $u = u(x, t)$ is a real valued function, which can for example model the evolution of the amplitude of a weakly nonlinear shallow dispersive wave in space and time (Korteweg & de Vries, 1895). The inputs $U(t) = U(u(t, \cdot))$ and $V(t) = V(u(t, \cdot))$ at the right end point of the boundary are feedbacks. The goal is to choose these boundary feedbacks so that solutions of (1.1) and (1.2) decay to zero as $t \rightarrow \infty$, at an exponential rate in the mean-square sense.

Controlling behavior of solutions has been one of the major subjects in the theory of evolution equations, and many approaches have been proposed. One method is to use local or global interior controllers. Another method is to use external (boundary) controllers, especially in those models where it is difficult to access the domain. Using feedback type controls is a common tactic to stabilize the solutions. However, non-feedback type controls (open loop control systems) are also used for steering solutions to or near a desired state. Exact, null, or approximate controllability models have been developed for almost all well-known PDEs by now.

Exact boundary controllability of the linear and nonlinear KdV equations with the same type of boundary conditions as in (1.1) was studied by (Rosier, 1997), (Coron & Crépeau, 2004), (Zhang, 1999), (Glass & Guerrero, 2008), (Cerpa, 2007), (Cerpa & Crépeau, 2009), (Rosier & Zhang, 2009), and (Glass & Guerrero, 2010). The main difference of these papers from our work is that in the exact control problem, the boundary inputs are chosen in advance to steer solutions to a desired final state at a given time. Hence, it is an open loop problem, whereas in our model the boundary inputs depend on the solution itself and therefore (1.1) is closed loop.

Stabilization of solutions of KdV equation with a localised interior damping was achieved by (Perla Menzala et al., 2002), (Pazoto, 2005), and (Massarolo et al., 2007). (Balogh & Krstic, 2000), There are also some results achieving stabilization of the KdV equation by using predetermined local boundary feedbacks, see for instance (W.-J. Liu & Krstić, 2002), and (Jia, 2016).

1.1. Motivation. (1.1) and (1.2) with homogeneous boundary conditions ($U = V \equiv 0$) are both dissipative since $\frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 \leq 0$. However, this does not always guarantee

the exponential decay. It is well-known that if $L \in \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N} \right\}$ (so

called *critical lengths* for KdV), then the solution does not need to decay to zero at all. For example if $L = 2\pi$, $u = 1 - \cos(x)$ is a (time independent) solution of (1.1) on $\Omega = (0, 2\pi)$ but its L^2 -norm is constant in t . On the other hand, if L is not critical, one can show the exponential stabilization of solutions for (1.1) under homogeneous boundary conditions, see for example (Perla Menzala et al., 2002, Theorem 2.1).

Recently, (Cerpa & Coron, 2013) studied the boundary feedback stabilization of the KdV equation with the boundary conditions

$$(1.3) \quad u(0, t) = U(t), u_x(L, t) = 0, u(L, t) = 0$$

by using the back-stepping technique, see e.g., (Krstic & Smyshlyaev, 2008) for this technique. (Cerpa & Coron, 2013) proved that given any $r > 0$, there corresponds a smooth kernel $k = k(x, y)$ such that the boundary feedback controller

$$U(t) = U(u(t, \cdot)) = \int_0^L k(0, y)u(y, t)dy$$

steers the solution of the linear KdV equation to zero with the decay rate estimate

$$\|u(t)\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} e^{-rt}.$$

Moreover, the same result also holds true for the nonlinear KdV equation provided that u_0 is sufficiently small in the L^2 -sense. Here, $k = k(x, y)$ is an appropriately chosen kernel function satisfying a third order PDE model on a triangular domain which involves three boundary conditions. In (Cerpa & Coron, 2013), the control acts on the Dirichlet boundary condition at the *left* end point of the domain. It is however stated that the situation is very different if the control acts at the *right* end point of the domain because then the kernel of the backstepping controller has to satisfy an overdetermined PDE model whose solution may or may not exist. Therefore, the problem of finding backstepping controllers acting at the *right* end point of the domain is interesting and the following problem remains open in their work:

Problem 1.4. *Is there a kernel $k = k(x, y)$ such that the solution of (1.1) and (1.2) satisfies*

$$(1.5) \quad \|u(\cdot, t)\|_{L^2(\Omega)} = \mathcal{O}(e^{-rt})$$

for some $r > 0$ with boundary feedback controllers given by

$$(1.6) \quad U(t) = \int_0^L k(L, y)u(y, t)dy \text{ and } V(t) = \int_0^L k_x(L, y)u(y, t)dy?$$

A stronger version of the above problem is the following:

Problem 1.7. *Given $r > 0$, can you find a kernel $k = k(x, y)$ such that the solution of (1.1) and (1.2) satisfies the L^2 -decay estimate (1.5) with the boundary feedback controllers given in (1.6)?*

This paper and the method proposed address only Problem 1.4, and the latter problem still remains open.

In order to understand the nature of the problem and the difficulty here, let us consider the linearised KdV equation in (1.1). A backstepping controller for this linear model is generally constructed by using a transformation given by

$$(1.8) \quad w(x, t) \equiv u(x, t) - \int_0^x k(x, y)u(y, t)dy,$$

where the unknown kernel function $k(x, y)$ is chosen in such a way that if u is a solution of (1.1) with the boundary feedback controllers given in (1.6), then w is a solution of the damped homogeneous initial-boundary value problem (so called “target system”)

$$(1.9) \quad \begin{cases} w_t + w_x + w_{xxx} + \lambda w = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ w(0, t) = w(L, t) = w_x(L, t) = 0 & \text{in } \mathbb{R}_+, \\ w(x, 0) = w_0(x) \equiv u_0 - \int_0^x k(x, y)u_0(y)dy & \text{in } \Omega. \end{cases}$$

The reason is that the solution of (1.9) satisfies $\|w(t)\|_{L^2(\Omega)} = O(e^{-\lambda t})$ and if the given transformation is invertible, one can hope to get a similar decay property for u .

The essence of the back-stepping algorithm is to be able to find an appropriate kernel function k which serves the purpose. In order to do this, one simply assumes that u solves (1.1) and plugs in $u(x, t) - \int_0^x k(x, y)u(y, t)dy$ into the main equation in (1.9) wherever one sees w . This gives a set of sufficient conditions that the kernel has to satisfy. Note that w satisfies the given homogeneous boundary conditions

$$w(0, t) = w(L, t) = w_x(L, t) = 0$$

by the transformation in (1.8) and the choice of the feedback controllers in (1.6). In order for the main equation in (1.9) to be satisfied, one can impose a few conditions on k . Indeed, computing the derivative of w with respect to t , we get

$$(1.10) \quad \begin{aligned} w_t(x, t) &= u_t(x, t) - \int_0^x u_t(y, t)k(x, y)dy \\ &= u_t(x, t) + \int_0^x u_y(y, t)k(x, y)dy + \int_0^x u_{yyy}(y, t)k(x, y)dy \\ &= u_t(x, t) + k(x, y)u(y, t)|_0^x - \int_0^x u(y, t)k_y(x, y)dy + k(x, y)u_{xx}(y, t)|_0^x - \int_0^x u_{yy}(y, t)k_y(x, y)dy \\ &= u_t(x, t) + k(x, x)u(x, t) - \int_0^x u(y, t)k_y(x, y)dy + k(x, x)u_{xx}(x, t) - k(x, 0)u_{xx}(0, t) \\ &\quad - u_x(y, t)k_y(x, y)|_0^x + \int_0^x u_y(y, t)k_{yy}(x, y)dy \\ &= u_t(x, t) + k(x, x)u(x, t) - \int_0^x u(y, t)k_y(x, y)dy + k(x, x)u_{xx}(x, t) - k(x, 0)u_{xx}(0, t) \\ &\quad - u_x(x, t)k_y(x, x) + k_y(x, 0)u_x(0, t) + u(y, t)k_{yy}(x, y)|_0^x - \int_0^x u(y, t)k_{yyy}(x, y)dy \\ &= u_t(x, t) + k(x, x)u(x, t) - \int_0^x u(y, t)k_y(x, y)dy + k(x, x)u_{xx}(x, t) \\ &\quad - k(x, 0)u_{xx}(0, t) - u_x(x, t)k_y(x, x) + k_y(x, 0)u_x(0, t) + u(x, t)k_{yy}(x, x) - \int_0^x u(y, t)k_{yyy}(x, y)dy. \end{aligned}$$

Now, computing the derivative of w with respect to x , we get

$$w_x(x, t) = u_x(x, t) - k(x, x)u(x, t) - \int_0^x k_x(x, y)u(y, t)dy.$$

Similarly, regarding the derivatives up to order three, we have

$$\begin{aligned} w_{xx}(x, t) &= u_{xx}(x, t) - k(x, x)u_x(x, t) - u(x, t)\frac{d}{dx}k(x, x) \\ &\quad - k_x(x, x)u(x, t) - \int_0^x k_{xx}(x, y)u(y, t)dy, \end{aligned}$$

and

$$\begin{aligned} w_{xxx}(x, t) &= u_{xxx}(x, t) - u(x, t)\frac{d^2}{dx^2}k(x, x) - 2u_x(x, t)\frac{d}{dx}k(x, x) \\ &\quad - u_{xx}(x, t)k(x, x) - u(x, t)\frac{d}{dx}k_x(x, x) - u_x(x, t)k_x(x, x) \\ &\quad - u(x, t)k_{xx}(x, x) - \int_0^x k_{xxx}(x, y)u(y, t)dy. \end{aligned}$$

Putting the above temporal and spatial derivatives together , we obtain the following:

$$\begin{aligned} (1.11) \quad & w_t(x, t) + w_x(x, t) + w_{xxx}(x, t) + \lambda w(x, t) = k_y(x, 0)u_x(0, t) \\ & - \int_0^x u(y, t) [k_{xxx}(x, y) + k_x(x, y) + k_{yyy}(x, y) + k_y(x, y) + \lambda k(x, y)] dy \\ & - k(x, 0)u_{xx}(0, t) - u_x(x, t) \left[k_y(x, x) + k_x(x, x) + 2\frac{d}{dx}k(x, x) \right] \\ & + u(x, t) \left[\lambda - k_{xx}(x, x) + k_{yy}(x, x) - \frac{d}{dx}k_x(x, x) - \frac{d^2}{dx^2}k(x, x) \right], \end{aligned}$$

The above equation is the same of the target system (1.9) if k solves the third order partial differential equation together with the set of boundary conditions given by

$$\begin{aligned} (1.12) \quad & k_{xxx} + k_{yyy} + k_y + k_x = -\lambda k, \\ & k(x, x) = k(x, 0) = k_y(x, 0) = 0, \\ & k_x(x, x) = \frac{\lambda}{3}x, \end{aligned}$$

where the PDE model is considered on the triangular spatial domain

$$\mathcal{T} \equiv \{(x, y) \in \mathbb{R}^2 \mid x \in [0, L], y \in [0, x]\} \text{ (see Figure 1 below).}$$

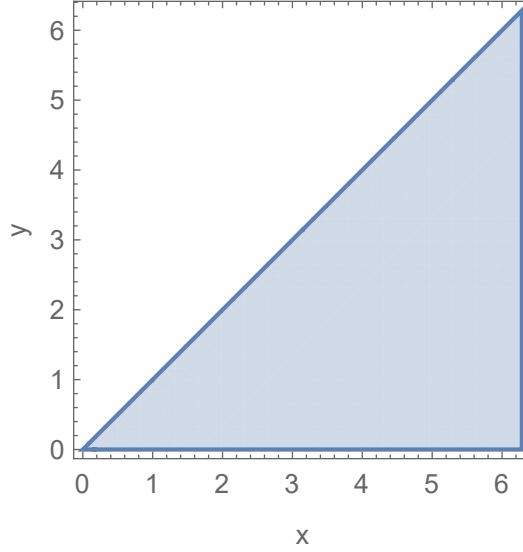


FIGURE 1. Triangular region T for $L = 2\pi$

In order to solve the problem (1.12), one generally applies a change of variables first. Here, an appropriate choice would be to define $t \equiv y$, $s \equiv x - y$, and $G(s, t) \equiv k(x, y)$. Then, G satisfies the boundary value problem given by

$$(1.13) \quad G_{ttt} - 3G_{s tt} + 3G_{s s t} + G_t = -\lambda G,$$

$$(1.14) \quad G(s, 0) = G_t(s, 0) = G(0, t) = 0,$$

$$(1.15) \quad G_s(0, t) = \frac{\lambda}{3}t$$

on the triangular domain

$$\mathcal{T}_0 \equiv \{(s, t) \mid t \in [0, L], s \in [0, L - t]\} \text{ (see Figure 2 below).}$$

Unfortunately, It is not easy to decide whether (1.13)-(1.15) has a solution. Note that, there is also a mismatch between the boundary conditions $G_t(s, 0) = 0$ and $G_s(0, t) = \frac{\lambda}{3}t$ in the sense that

$$G_{ts}(0, 0) = 0 \neq G_{st}(0, 0) = \frac{\lambda}{3}.$$

Hence, the standard back-stepping algorithm fails because it enforces us to solve an overdetermined singular PDE model. This issue does not appear if one tries to control the system from the left endpoint of the domain as in (Cerpa & Coron, 2013).

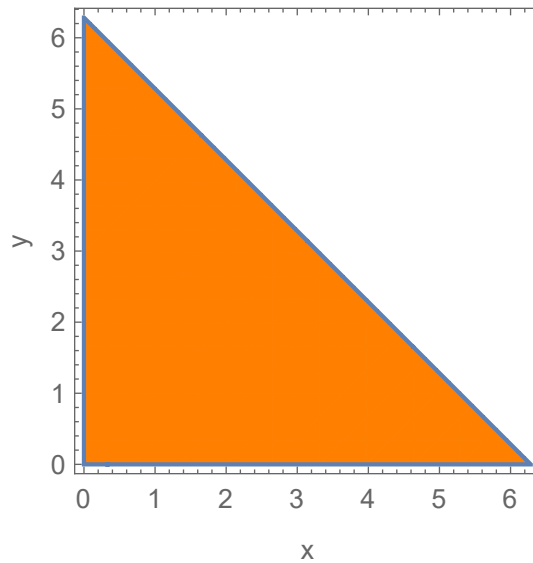


FIGURE 2. Triangular region T_0 for $L = 2\pi$

1.2. Pseudo-backstepping. We introduce a new backstepping technique which eliminates the difficulties explained in the previous section. In the standard backstepping method, at first the plant model (1.1) is transformed into the most desirable (e.g., exponentially stable) target system with a transformation as in (1.8). This is called the forward transformation. Secondly, the target system is transformed back into the plant model via an inverse transformation, generally in the form

$$(1.16) \quad u(x,t) = w(x,t) + \int_0^x p(x,y)w(y,t)dy.$$

This is called the backward transformation. A combination of these two steps allows one to conclude that the plant is stable if and only if the target system is stable in the same sense (see Figure 3).

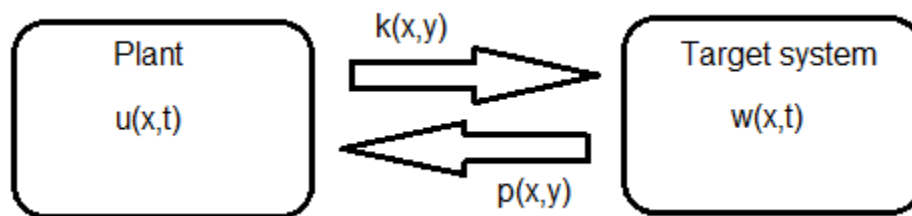


FIGURE 3. Standard back-stepping

Unfortunately, applying this algorithm to our problem forces the kernels p and k to be solutions of overdetermined boundary value problems and the method fails.

Our strategy is based on using a pseudo kernel which is chosen as a solution of a corrected version of the gain control PDE given by:

$$(1.17) \quad \tilde{G}_{ttt} - 3\tilde{G}_{stt} + 3\tilde{G}_{sst} + \tilde{G}_t = -\lambda\tilde{G},$$

$$(1.18) \quad \tilde{G}(s, 0) = \tilde{G}(0, t) = 0,$$

$$(1.19) \quad \tilde{G}_s(0, t) = \frac{\lambda}{3}t$$

on the triangular domain \mathcal{T}_0 . The difference of this model than the previous one in (1.13)-(1.15) is that the boundary condition $\tilde{G}_t(s, 0) = 0$ is completely disregarded. The advantage of using this modified model is though, first of all we can solve it and secondly, although the boundary condition $\tilde{G}_t(s, 0) = 0$ is disregarded in the model, we can control the size of this boundary condition by choosing λ sufficiently small. The cost of using a pseudo kernel is that the target system changes (see the modified target system in (2.17)) and this causes a slower rate of decay. Nevertheless, this new method (will be referred as *pseudo-backstepping* from now on) allows us to obtain physically reasonable exponential decay rates for some choice of λ (see Table 1 for sample decay rates for some values of λ on a domain of length $L = 2\pi$).

Another ingredient of our method is that instead of using a concrete backward transformation as in (1.16), we rely on the existence of an abstract inverse transformation which maps the solution of the modified target system back into the original plant. The existence of such transformation is proved via succession (see Lemma 2.22 below). This type of backward transformation was used previously within the context of the stabilization of the heat equation with a localized source of instability (W. Liu, 2003). The reason for not searching for an inverse of type (1.16) is because otherwise one encounters again a highly overdetermined system by computing the temporal and spatial derivatives of the given transformation and finding the conditions that p has to satisfy.

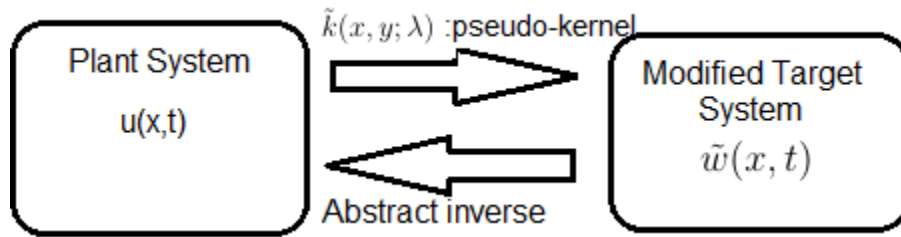


FIGURE 4. Pseudo-backstepping

1.3. Main results. Applying the pseudo-backstepping method explained above to the linearized and nonlinear KdV models given in (1.1) and (1.2), we are able to prove the following wellposedness and stabilization theorems:

Theorem 1.20 (Wellposedness). *Let $T > 0$, $u_0 \in L^2(\Omega)$ and*

$$(1.21) \quad U(t) = \int_0^L \tilde{k}(L, y)u(y, t)dy, \quad V(t) = \int_0^L \tilde{k}_x(L, y)u(y, t)dy,$$

where \tilde{k} is a smooth kernel given by (2.15). Then, (1.1) has a unique solution

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

satisfying also

$$u_x \in C([0, L]; L^2(0, T)).$$

Moreover, the same result also holds true for the nonlinear KdV equation (1.2) if $\|u_0\|_{L^2(\Omega)}$ is sufficiently small.

Remark 1.22. Indeed, our analysis in this paper also shows that if $u_0 \in H^3(\Omega)$ and it satisfies the compatibility conditions

$$(1.23) \quad u_0(0) = 0, u_0(L) = \int_0^L \tilde{k}(L, y) u_0(y) dy, u_0'(L) = \int_0^L \tilde{k}_x(L, y) u_0(y) dy,$$

then the solution of (1.1) or the local solution of (1.2) satisfies

$$u \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)).$$

One can also interpolate to get regularity in the fractional spaces. For example, let $u_0 \in H^s(\Omega)$ ($s \in [0, 3]$) so that it satisfies the compatibility conditions

$$u_0(0) = 0, u_0(L) = \int_0^L \tilde{k}(L, y) u_0(y) dy$$

if $s \in [0, 3/2]$ and the compatibility conditions (1.23) if $s \in (3/2, 3]$. Then, the solution of (1.1) or the local solution of (1.2) satisfies

$$u \in C([0, T]; H^s(\Omega)) \cap L^2(0, T; H^{s+1}(\Omega)).$$

Theorem 1.24 (Stabilization). Let $u_0 \in L^2(\Omega)$. Then, for sufficiently small $\lambda > 0$, one has $\alpha = \lambda - \frac{1}{2} \|\tilde{k}_y(\cdot, 0)\|_{L^2(\Omega)}^2 > 0$, where \tilde{k} is given by (2.15), and the corresponding solution of (1.1) with the boundary feedback controllers (1.21) satisfies

$$\|u(t)\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} e^{-\alpha t}.$$

Moreover, the same decay property is also true for the nonlinear KdV equation (1.2) if $\|u_0\|_{L^2(\Omega)}$ is sufficiently small.

Remark 1.25. The proof of Theorem 1.24 is given in the next section. Table 1 gives some examples where the exponential stabilization can be achieved. For example, when $\lambda = 0.03$, the decay rate is approximately of order $\mathcal{O}(e^{-0.18t})$ on a domain of length $L = 2\pi$. The exponential decay rate is substantially small (see Table 1) relative to the decay rates one can get by controlling the equation from the left end-point with the same type of boundary conditions. Indeed, the important thing is not where the controller is located, what matters is the number of boundary conditions specified on the opposite side of the boundary. For example, if one specified two boundary condition at the left and only one boundary condition at the right, then it would be easier to control from right and more difficult to control from left as opposed to the problem studied in this paper.

λ	$\alpha = \lambda - \frac{1}{2} \ \tilde{k}_y(\cdot, 0)\ _{L^2(\Omega)}^2$
0.01	0.00954938
0.02	0.0167563
0.03	0.0181985
0.04	0.00844268
0.05	-0.0203987
0.10	-0.961935
1	-83925.8

TABLE 1. Numerical experiments on a domain of critical length $L = 2\pi$

2. STABILIZATION

In this section, we prove Theorem 1.24. At first, we prove the existence of the pseudo-kernel and the abstract inverse transformation. Secondly, by using the multiplier method, we obtain the stabilization for suitable λ . The multiplier method is applied only formally, but the calculations can be justified by a standard density argument the regularity results proved in the next section.

2.1. Linearised model. The sought-after solution of (1.17)-(1.19) can be constructed by applying the successive approximations technique to the integral equation

$$(2.1) \quad \tilde{G}(s, t) = \frac{\lambda}{3}st + \frac{1}{3} \int_0^t \int_0^s \int_0^\omega (-\tilde{G}_{ttt} + 3\tilde{G}_{stt} - \tilde{G}_t - \lambda\tilde{G})(\xi, \eta) d\xi d\omega d\eta.$$

Indeed, we have the following lemma.

Lemma 2.2. *There exists a C^∞ -function \tilde{G} such that \tilde{G} solves the integral equation (2.1) as well as the boundary value problem given in (1.17)-(1.19).*

Proof. Let P be defined by

$$(2.3) \quad (Pf)(s, t) = \frac{1}{3} \int_0^t \int_0^s \int_0^\omega (-f_{ttt} + 3f_{stt} - f_t - \lambda f)(\xi, \eta) d\xi d\omega d\eta.$$

By (2.1), we need to solve the following equation,

$$\tilde{G}(s, t) = \frac{\lambda}{3}st + P\tilde{G}(s, t).$$

Define $\tilde{G}^0 \equiv 0$, $\tilde{G}^1(s, t) = \frac{\lambda}{3}st$, and $\tilde{G}^{n+1} = \tilde{G}^1 + P\tilde{G}^n$. Then for $n \geq 1$,

$$\tilde{G}^{n+1} - \tilde{G}^n = P(\tilde{G}^n - \tilde{G}^{n-1}).$$

So if we define $H^0(s, t) = st$ and $H^{n+1} = PH^n$, we get $H^n = \frac{3}{\lambda}(\tilde{G}^{n+1} - \tilde{G}^n)$. Moreover, for $j > i$,

$$(2.4) \quad \tilde{G}^j - \tilde{G}^i = \sum_{n=i}^{n=j-1} \tilde{G}^{n+1} - \tilde{G}^n = \frac{\lambda}{3} \sum_{n=i}^{n=j-1} H^n.$$

Let $\|\cdot\|_\infty$ denote the supremum norm of a function on the triangle T_0 . By (2.4) in order to show \tilde{G}_n (and its partial derivatives) is Cauchy with respect to the norm $\|\cdot\|_\infty$ it is

enough to show H^n (and its partial derivatives) is an absolutely summable sequence with respect to the same norm.

To show H^n 's are absolutely summable let us first write P as sum of four operators

$$P = P_{-2} + P_{-1} + P_0 + P_1,$$

where

$$P_{-2}f = \frac{1}{3} \int_0^t \int_0^s \int_0^\omega -f_{ttt}(\xi, \eta) d\xi d\omega' d\eta,$$

$$P_{-1}f = \int_0^t \int_0^s \int_0^\omega f_{stt}(\xi, \eta) d\xi d\omega' d\eta,$$

$$P_0f = \frac{1}{3} \int_0^t \int_0^s \int_0^\omega -f_t(\xi, \eta) d\xi d\omega' d\eta,$$

$$P_1f = \frac{1}{3} \int_0^t \int_0^s \int_0^\omega -\lambda f(\xi, \eta) d\xi d\omega' d\eta.$$

Then

$$(2.5) \quad H^n = P^n H^0 = (P_{-2} + P_{-1} + P_0 + P_1)^n st = \sum_{r=1}^{4^n} R_{r,n} st$$

where $R_{r,n} := P_{j_{r,n}} P_{j_{r,n-1}} \cdots P_{j_{r,1}}$, $j_{r,i} \in \{-2, -1, 0, 1\}$.

Observe that for positive integers m and nonnegative integers k

$$(2.6) \quad P_{-1} s^m t^k = c_{-1} s^{m+1} t^{k-1} \text{ and } P_i s^m t^k = c_i s^{m+2} t^{k+i} \text{ for } i = -2, 0, 1,$$

where

$$(2.7) \quad c_{-2} = \begin{cases} 0 & \text{if } k \leq 2, \\ -\frac{k(k-1)}{3(m+1)(m+2)} & \text{if } k > 2, \end{cases}$$

$$(2.8) \quad c_{-1} = \begin{cases} 0 & \text{if } k \leq 1, \\ \frac{k}{(m+1)} & \text{if } k > 1, \end{cases}$$

$$(2.9) \quad c_0 = -\frac{1}{3(m+1)(m+2)},$$

$$(2.10) \quad c_1 = -\frac{\lambda}{3(m+1)(m+2)(k+1)}.$$

Let $\sigma = \sigma(n, r) = \sum_{i=1}^n j_{r,i}$. From (2.6)-(2.10) one can easily see that for each n and r

$$(2.11) \quad R_{r,n} st = \begin{cases} 0 & \text{if } \sigma < -1, \\ C_{r,n} s^\beta t^{\sigma+1} & \text{if } \sigma \geq -1 \end{cases}$$

where $n+1 \leq \beta \leq 2n+1$ and $C_{r,n}$ is a constant which only depends on n and r .

Let $\tilde{\lambda} = \max\{1, \lambda\}$. We claim that for each n and r ,

$$(2.12) \quad |C_{r,n}| \leq \frac{\tilde{\lambda}^n}{(n+1)!(\sigma+1)!}.$$

Taking $m = 1, k = 1$ in (2.6)-(2.10) one can check that the claim holds for $n = 1$. Suppose it holds for $n = \ell - 1$ and for all $r \in \{1, 2, \dots, 4^{\ell-1}\}$. Then for $n = \ell$ and $r^* \in \{1, 2, \dots, 4^\ell\}$, using (2.6) and (2.11), we obtain

$$R_{r^*, \ell} s t = P_i R_{r, \ell-1} s t = C_{r, \ell-1} P_i s^\beta t^{\sigma+1} = C_{r, \ell-1} c_i s^{\beta^*} t^{\sigma^*+1}$$

for some $i \in \{-2, -1, 0, 1\}$ and $r \in \{1, 2, \dots, 4^{\ell-1}\}$, where β^* is either $\beta + 1$ or $\beta + 2$, $\sigma^* = \sigma + i$. By the induction assumption

$$C_{r, \ell-1} \leq \frac{\tilde{\lambda}^{\ell-1}}{\ell!(\sigma+1)!}.$$

Moreover (2.7)-(2.10) and the fact that $\beta \geq \ell$ imply $|c_i| \leq \frac{\sigma+1}{\ell+1}$ for $i = -1, -2$, $|c_0| < \frac{1}{\ell+1}$, and $|c_1| < \frac{\lambda}{(\sigma+2)(\ell+1)}$. Hence for each $i \in \{-2, -1, 0, 1\}$ we get

$$|C_{r^*, \ell}| = |C_{r, (\ell-1)} c_i| \leq \frac{\tilde{\lambda}^\ell}{(\ell+1)!(\sigma+i+1)!} = \frac{\tilde{\lambda}^\ell}{(\ell+1)!(\sigma^*+1)!}$$

which proves that the claim holds for $n = \ell$ as well.

By (2.5), (2.11), (2.12) and the fact that $0 \leq s, t \leq L$ in the triangle T_0 , we obtain

$$(2.13) \quad \|H^n\|_\infty \leq \frac{4^n \tilde{\lambda}^n L^{3n+2}}{(n+1)!}$$

which is summable. Moreover since H^n is a linear combination of 4^n monomials of the form $s^\beta t^{\sigma+1}$ with $\beta \leq 2n+1$ and $\sigma \leq n$, any partial derivative $\partial_s^a \partial_t^b H^n$ of H^n will be absolutely less than

$$(2.14) \quad \frac{(2n+1)^a (n+1)^b 4^n \tilde{\lambda}^n L^{3n+2-a-b}}{(n+1)!}$$

which is also summable. □

Now, we define the pseudo-kernel by

$$(2.15) \quad \tilde{k}(x, y) := \tilde{G}(x - y, y)$$

and consider the transformation given by

$$(2.16) \quad \tilde{w}(x, t) \equiv u(x, t) - \int_0^x \tilde{k}(x, y) u(y, t) dy.$$

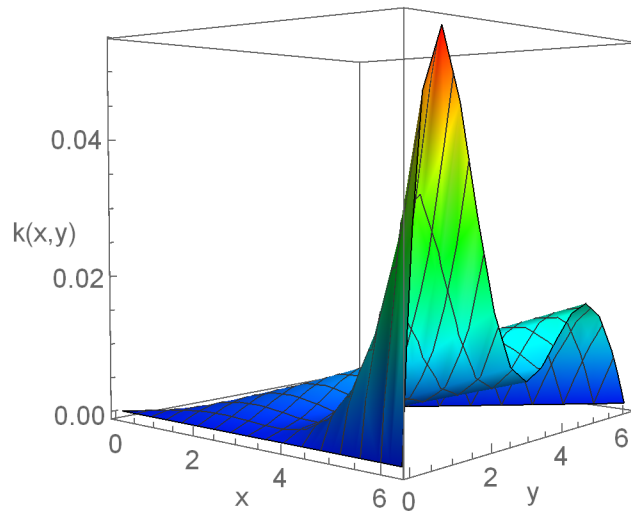


FIGURE 5. Pseudo kernel \tilde{k} when $\lambda = 0.01$ ($L = 2\pi$)

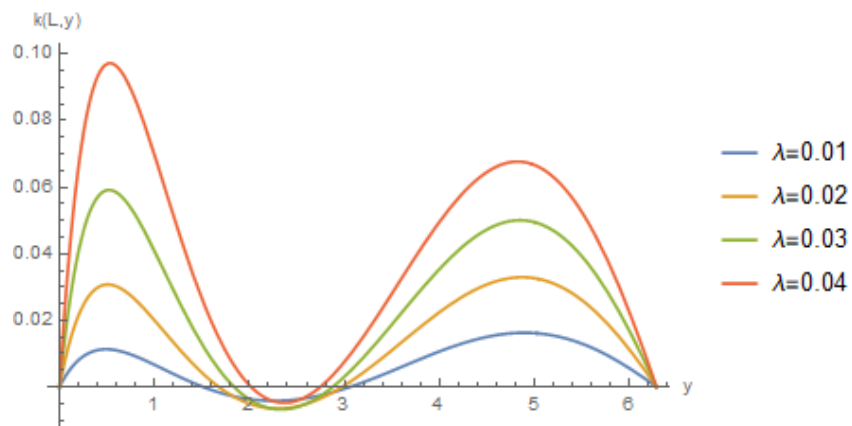


FIGURE 6. Control effort at the Dirichlet b.c. for different λ ($L = 2\pi$)

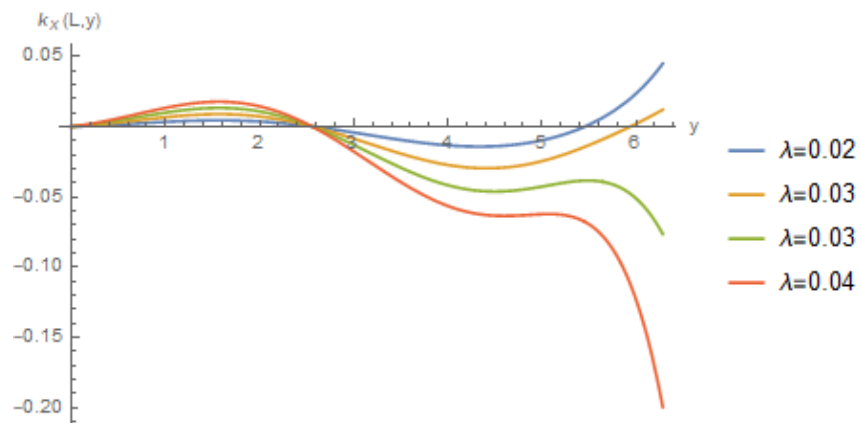


FIGURE 7. Control effort at the Neumann b.c. for different λ ($L = 2\pi$)

Note that we have $\tilde{u}_x(0, t) = \tilde{w}_x(0, t)$ by the boundary conditions of \tilde{k} . Using this fact, we can rewrite the modified target system as

$$(2.17) \quad \begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = \tilde{k}_y(x, 0) \tilde{w}_x(0, t) & \text{in } \Omega \times \mathbb{R}_+, \\ \tilde{w}(0, t) = \tilde{w}(L, t) = \tilde{w}_x(L, t) = 0 & \text{in } \mathbb{R}_+, \\ \tilde{w}(x, 0) = \tilde{w}_0(x) := u_0 - \int_0^x \tilde{k}(x, y) u_0(y) dy & \text{in } \Omega. \end{cases}$$

Multiplying the above model by \tilde{w} and integrating over $(0, 1)$, using the Cauchy-Schwarz inequality, we obtain

$$(2.18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{w}(t)\|_{L^2(\Omega)}^2 + \lambda \|\tilde{w}(t)\|_{L^2(\Omega)}^2 \\ \leq -\frac{1}{2} |\tilde{w}_x(0, t)|^2 + \int_0^L \tilde{k}_y(x, 0) \tilde{w}_x(0, t) \tilde{w}(x, t) dx \\ \leq \cancel{-\frac{1}{2} |\tilde{w}_x(0, t)|^2} + \frac{1}{2} |\tilde{w}_x(0, t)|^2 + \frac{1}{2} \left(\int_0^L |\tilde{k}_y(x, 0)| |\tilde{w}(x, t)| dx \right)^2. \end{aligned}$$

Since \tilde{k} is smooth on the compact set \mathcal{T} , we have

$$(2.19) \quad \frac{1}{2} \frac{d}{dt} \|\tilde{w}(t)\|_{L^2(\Omega)}^2 + \left(\lambda - \frac{1}{2} \|\tilde{k}_y(\cdot, 0)\|_{L^2(\Omega)}^2 \right) \|\tilde{w}(t)\|_{L^2(\Omega)}^2 \leq 0.$$

It follows that

$$(2.20) \quad \|\tilde{w}(t)\|_{L^2(\Omega)}^2 \leq \|\tilde{w}_0\|_{L^2(\Omega)}^2 e^{-2\alpha t},$$

where

$$\alpha \equiv \lambda - \frac{1}{2} \|\tilde{k}_y(\cdot, 0)\|_{L^2(\Omega)}^2.$$

The graph of the function $\tilde{k}_y(\cdot, 0)$ is depicted in Figure 8 on a domain of length $L = 2\pi$.

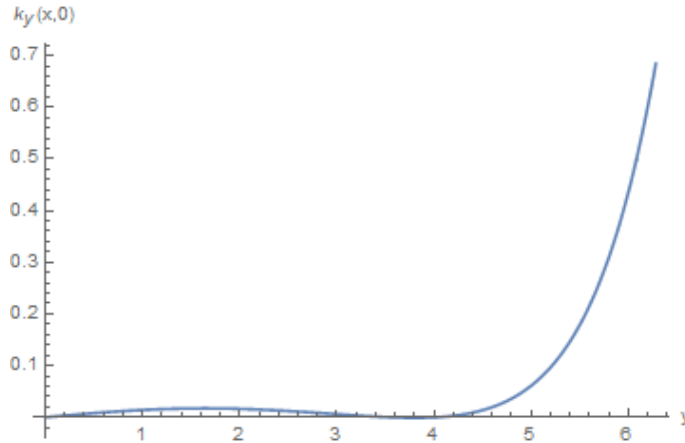


FIGURE 8. Pseudo kernel \tilde{k} when $\lambda = 0.03$ ($L = 2\pi$)

By taking $L^2(\Omega)$ norms of both sides of (2.16) (with $t = 0$) and using the Cauchy-Schwarz inequality, we get

$$(2.21) \quad \|\tilde{w}_0\|_{L^2(\Omega)} \leq \left(1 + \|\tilde{k}\|_{L^2(\mathcal{T})}\right) \|u_0\|_{L^2(\Omega)}.$$

Let $K : H^l(\Omega) \rightarrow H^l(\Omega)$ ($l \geq 0$) be the integral operator defined by

$$(K\varphi)(x) := \int_0^x \tilde{k}(x, y)\varphi(y)dy.$$

It is not difficult to prove that the operator $I - K$ is invertible from $H^l(\Omega) \rightarrow H^l(\Omega)$ (for $l \geq 0$) with a bounded inverse. This is proved in a general setting in the lemma below:

Lemma 2.22. *$I - K$ is invertible with a bounded inverse from $H^l(\Omega) \rightarrow H^l(\Omega)$ ($l \geq 0$).*

Remark 2.23. *The above lemma can be expressed in a sharper form. Indeed, the proof below shows that Φ is a bounded operator from $L^2(\Omega) \rightarrow H^l(\Omega)$ ($l = 0, 1, 2$) and it is a bounded operator from $H^{l-2}(\Omega) \rightarrow H^l(\Omega)$ ($l > 2$).*

Proof. The above lemma can be proved by slightly modifying the proof of (W. Liu, 2003, Lemma 2.4). However, we will still give a brief proof here since we will crucially need to refer to some ingredients of the proof of this lemma later in the proofs of the stabilization and well-posedness results.

To this end, let us first consider the case $l = 0$ and let $\psi = (I - K)\varphi$ for some $\varphi \in L^2(\Omega)$. The idea is to first write $\psi = \varphi - v$ where $v = K\varphi$. Note that then,

$$\psi(x) = \varphi(x) - [K\varphi](x) = (\psi(x) + v(x)) - \int_0^x \tilde{k}(x, y)(\psi(y) + v(y))dy.$$

This gives

$$v(x) = \int_0^x \tilde{k}(x, y)\psi(y)dy + \int_0^x \tilde{k}(x, y)v(y)dy.$$

Given a fixed ψ , one can solve this equation via succession (see (W. Liu, 2003, Lemma 2.4) for the details of the succession argument). This implicitly defines a linear operator $\Phi : \psi \mapsto v$ on $L^2(\Omega)$ with the property that Φ is bounded, i.e., there exists $C_0 > 0$ such that

$$(2.24) \quad \|v\|_{L^2(\Omega)} \leq C_0\|\psi\|_{L^2(\Omega)},$$

where C_0 depends only on $\|\tilde{k}\|_{L^\infty(\mathcal{T})}$. But then, φ is simply equal to $(I + \Phi)\psi$ and therefore $(I - K)^{-1}$ exists and equal to $I + \Phi$, moreover it is bounded. By differentiating and using the smoothness of \tilde{k} , $(I - K)^{-1}$ extends to a linear bounded operator also on Sobolev spaces $H^l(\Omega)$ ($l \geq 1$). Indeed, since $\tilde{k}(x, x) = 0$, we have

$$(2.25) \quad v_x(x) = \int_0^x \tilde{k}_x(x, y)(\psi(y) + v(y))dy,$$

which implies

$$(2.26) \quad \|v_x\|_{L^2(\Omega)} \leq \|\tilde{k}_x\|_{L^2(\mathcal{T})} \left(\|\psi\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \right).$$

Hence, using (2.24), we have

$$(2.27) \quad \|v\|_{H^1(\Omega)} \leq C_1\|\psi\|_{L^2(\Omega)},$$

where C_1 depends on $\|\tilde{k}_x\|_{L^2(\mathcal{T})}$ and C_0 . This shows that Φ is bounded from $L^2(\Omega)$ into $H^1(\Omega)$, a fortiori bounded from $H^1(\Omega)$ into $H^1(\Omega)$. Now for $l = 2$, using $k_x(x, x) = \frac{\lambda}{3}x$,

$$(2.28) \quad (\partial_x^2 v)(x) = \frac{\lambda}{3}x(\psi(x) + v(x)) + \int_0^x (\partial_x^2 \tilde{k})(x, y)(\psi(y) + v(y))dy.$$

Taking $L^2(\Omega)$ norms of both sides and using the previous inequalities, we get

$$(2.29) \quad \|v\|_{H^2(\Omega)} \leq C_2 \|\psi\|_{L^2(\Omega)},$$

where C_2 depends on $\|\partial_x^2 \tilde{k}\|_{L^2(\mathcal{T})}$, C_1 , and λ . This shows that Φ is bounded from $L^2(\Omega)$ into $H^2(\Omega)$, a fortiori bounded from $H^1(\Omega)$ or $H^2(\Omega)$ into $H^2(\Omega)$. Going in the same fashion, one can show that

$$(2.30) \quad \|v\|_{H^3(\Omega)} \leq C_3 \|\psi\|_{H^1(\Omega)},$$

where C_3 is a fixed constant depending on various norms of \tilde{k} . More generally,

$$(2.31) \quad \|v\|_{H^l(\Omega)} \leq C_l \|\psi\|_{H^{l-2}(\Omega)},$$

where $l > 2$ and C_l depends on various norms of \tilde{k} . Hence, for $l > 2$, Φ is a bounded operator from $H^{l-2}(\Omega)$ into $H^l(\Omega)$, and a fortiori bounded from $H^l(\Omega)$ into $H^l(\Omega)$. \square

Remark 2.32. Another important estimate that follows from (2.25) via (2.24) is that

$$(2.33) \quad \|v_x\|_{L^\infty(\Omega)} \leq C \|\psi\|_{L^2(\Omega)}$$

for some $C > 0$ which depends on \tilde{k} .

From the above lemma, it follows in particular that $u(x, t) = [(I - K)^{-1}\tilde{w}](x, t)$, and moreover

$$(2.34) \quad \|u(t)\|_{L^2(\Omega)} \leq \|(I - K)^{-1}\|_{B[L^2(\Omega)]} \cdot \|\tilde{w}(t)\|_{L^2(\Omega)},$$

where $\|\cdot\|_{B[L^2(\Omega)]}$ is the operator norm of $(I - K)^{-1}$ from $L^2(\Omega)$ into $L^2(\Omega)$.

Combining (2.34) with (2.20) and (2.21), we conclude that

$$(2.35) \quad \|u(t)\|_{L^2(\Omega)} \leq \left(1 + \|\tilde{k}\|_{L^2(\mathcal{T})}\right) \|(I - K)^{-1}\|_{B[L^2(\Omega)]} \|u_0\|_{L^2(\Omega)} e^{-\alpha t}.$$

We can prove that the parameter α in the above estimate is positive if λ is sufficiently small. Indeed, we have the following lemma.

Lemma 2.36. For a given L , there exists sufficiently small λ such that

$$\alpha = \lambda - \frac{1}{2}\|k_y(\cdot, 0)\|_{L^2}^2 > 0.$$

Proof. Taking the partial derivative of both sides of (2.4) with respect to t and taking $i = 0$ we see that

$$\tilde{G}_t^j(s, t) = \frac{\lambda}{3} \sum_{n=0}^{j-1} H_t^n(s, t).$$

Passing to the limit we obtain

$$\tilde{G}_t(s, t) = \frac{\lambda}{3} \sum_{n=0}^{\infty} H_t^n(s, t).$$

Note that for $\lambda < 1$, $\tilde{\lambda} = 1$. Therefore by (2.13) the summation term is absolutely less than some constant M which only depends on L . Hence we get

$$\|\tilde{G}_t\|_\infty \leq \frac{\lambda M}{3}.$$

Since $k_y(x, 0) = \tilde{G}_t(s, 0)$, in particular we have

$$\|k_y(\cdot, 0)\|_{L^2}^2 \leq L \|k_y(\cdot, 0)\|_\infty^2 \leq L \|\tilde{G}_t\|_\infty^2 \leq \frac{\lambda^2 M^2 L}{9}.$$

As a result

$$\alpha = \lambda - \frac{1}{2} \|k_y(\cdot, 0)\|_{L^2}^2 \geq \lambda - \frac{\lambda^2 M^2 L}{18} = \lambda^2 \left(\frac{1}{\lambda} - \frac{M^2 L}{18} \right)$$

which is positive for sufficiently small λ . \square

The inequality (2.35) together with Lemma 2.36 proves the linear part of Theorem 1.24.

2.2. Nonlinear model. In this section, we consider the nonlinear KdV model (1.2) with the feedback controllers given in (1.21). By using the transformation given in (2.16), we obtain the following PDE from (1.1), noting that $\tilde{k}(x, x) = 0$:

$$(2.37) \quad \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \lambda \tilde{w} = \tilde{k}_y(\cdot, 0) \tilde{w}_x(0, \cdot) - (\tilde{w} + v) (\tilde{w}_x + v_x)$$

with homogeneous boundary conditions

$$(2.38) \quad \tilde{w}(0, t) = 0, \quad \tilde{w}(L, t) = 0, \quad \text{and} \quad \tilde{w}_x(L, t) = 0,$$

where $v(x, t) = [\Phi \tilde{w}](x, t)$ with Φ being the linear operator defined in Section 2.1 in the proof of Lemma 2.22. Multiplying (2.37) by $\tilde{w}(x, t)$ and integrating over $\Omega = (0, L)$, we obtain

$$(2.39) \quad \begin{aligned} \int_0^L \tilde{w}(x, t) \tilde{w}_t(x, t) dx &= \int_0^L \tilde{k}_y(x, 0) \tilde{w}_x(0, t) \tilde{w}(x, t) dx - \int_0^L \tilde{w}(x, t) \tilde{w}_x(x, t) dx \\ &- \int_0^L \tilde{w}(x, t) \tilde{w}_{xxx}(x, t) dx - \lambda \int_0^L \tilde{w}^2(x, t) dx - \int_0^L \tilde{w}^2(x, t) \tilde{w}_x(x, t) dx - \int_0^L \tilde{w}^2(x, t) v_x(x, t) dx \\ &- \int_0^L \tilde{w}(x, t) \tilde{w}_x(x, t) v(x, t) dx - \int_0^L \tilde{w}(x, t) v(x, t) v_x(x, t) dx. \end{aligned}$$

Using integration by parts and Cauchy-Schwarz inequality, it follows that

$$(2.40) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{w}(t)\|_{L^2(\Omega)}^2 + \left(\lambda - \frac{1}{2} \|\tilde{k}_y(\cdot, 0)\|_{L^2(\Omega)}^2 \right) \|\tilde{w}(t)\|_{L^2(\Omega)}^2 \\ \leq \frac{3}{2} \|\tilde{w}(t)\|_{L^2(\Omega)}^2 \|v_x(t)\|_{L^\infty(\Omega)} + \|\tilde{w}(t)\|_{L^2(\Omega)} \|v(t)\|_{L^2(\Omega)} \|v_x(t)\|_{L^\infty(\Omega)}. \end{aligned}$$

Using (2.24) and (2.33), we deduce the following inequality:

$$(2.41) \quad y' + 2\alpha y - cy^{\frac{3}{2}} \leq 0,$$

where $y(t) \equiv \|\tilde{w}(t)\|_{L^2(\Omega)}^2$ and c is a constant which depends on \tilde{k} . Solving the inequality (2.41) and assuming $\|\tilde{w}_0\|_{L^2(\Omega)} < \frac{\alpha}{c}$, we get

$$(2.42) \quad \|\tilde{w}(t)\|_{L^2(\Omega)}^2 = y(t) \leq \frac{1}{\left[\left(\frac{1}{\|\tilde{w}_0\|_{L^2(\Omega)}} - \frac{c}{2\alpha}\right)e^{\alpha t} + \frac{c}{2\alpha}\right]^2} < \frac{1}{\left[\frac{e^{\alpha t}}{2\|\tilde{w}_0\|_{L^2(\Omega)}}\right]^2}.$$

Recall that $\|\tilde{w}_0\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)}$. Combining this with (2.34) and (2.42), we deduce

$$(2.43) \quad \|u(t)\|_{L^2(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} e^{-\alpha t}, \text{ for } t \geq 0.$$

Hence, the proof of Theorem 1.24 for the nonlinear KdV equation is also complete. Note that the smallness assumption on the initial datum \tilde{w}_0 implies a smallness assumption on u_0 due to the fact that we also have $\|u_0\|_{L^2(\Omega)} \lesssim \|\tilde{w}_0\|_{L^2(\Omega)}$ thanks to Lemma 2.22.

3. WELL-POSEDNESS

In this section, we prove the well-posedness of the PDE models studied in the previous sections. For simplicity, we assume $L = 1$ throughout this section. This assumption has no consequence as far as the wellposedness is concerned, and all results proved here are also true for any $L > 0$. Thanks to the Lemma 2.2, it is enough to prove the well-posedness of the respective modified target systems in order to obtain well-posedness of (1.1) and (1.2).

3.1. Linearised model. Consider the following linear KdV equation with homogeneous boundary conditions.

$$(3.1) \quad \begin{cases} y_t + y_x + y_{xxx} + \lambda y = a(x)y_x(0, \cdot) & \text{in } \Omega \times \mathbb{R}_+, \\ y(0, t) = y(1, t) = y_x(1, t) = 0 & \text{in } \mathbb{R}_+, \\ y(x, 0) = y_0 \in L^2(\Omega) & \text{in } \Omega. \end{cases}$$

We have the following result.

Proposition 3.2. *i) Let $T' > 0$ be arbitrary and $y_0, a \in L^2(\Omega)$. Then, there exists $T \in (0, T')$ independent of the size of y_0 such that (3.1) has a unique local solution*

$$y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

satisfying also

$$y_x \in C([0, 1]; L^2(0, T)).$$

Moreover, if $a \in L^\infty(\Omega)$, then y extends as a global solution. In other words, T can be taken as T' .

ii) Let $a \in H^1(\Omega)$ and $y_0 \in H^3(\Omega)$ satisfying the computability conditions

$$y_0(0) = y_0(1) = y_0'(1) = 0.$$

Then, the (local/global) solution in part (i) enjoys the extra regularity

$$y \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)).$$

Proof. Step 1 - Local wellposedness: Let us define the linear operator

$$A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$$

by

$$A\varphi := -\varphi' - \varphi''',$$

where

$$D(A) := \{\varphi \in H^3(\Omega) : \varphi(0) = \varphi(1) = \varphi'(1) = 0\}.$$

Then, the initial boundary value problem (3.1) can be rewritten in the abstract operator theoretic form

$$(3.3) \quad \begin{cases} \dot{y} = Ay + Fy, \\ y(0) = y_0, \end{cases}$$

where $F\varphi := -\lambda\varphi + a(\cdot)\gamma_1^0\varphi$. Here, γ_1^0 is the first order trace operator at the left end-point, i.e., $\gamma_1^0\varphi := \varphi'(0)$. This operator is well-defined for $\varphi \in H^{\frac{3}{2}+\epsilon}(\Omega) \supset D(A)$.

It is not difficult to see that the adjoint of A is defined by

$$A^*\varphi := \varphi' + \varphi'''$$

with

$$D(A^*) := \{\varphi \in H^3(\Omega) : \varphi(0) = \varphi(1) = \varphi'(0) = 0\}.$$

A is a densely defined closed operator, and moreover, A and A^* are dissipative (Rosier, 1997, Proposition 3.1). Therefore, A generates a strongly continuous semigroup of contractions $\{S(t)\}_{t \geq 0}$ on $L^2(\Omega)$ (Pazy, 1983, Corollary I.4.4). Now we construct the operator

$$(3.4) \quad y = [\Psi z](t) := S(t)y_0 + \int_0^t S(t-s)Fz(s)ds.$$

Let us define the space (see e.g., (Bona et al., 2003))

$$(3.5) \quad Y_T := \{z \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \mid z_x \in C([0, 1]; L^2(0, T))\}$$

equipped with the norm

$$\|z\|_{Y_T} := \left(\|z\|_{C([0, T]; L^2(\Omega))}^2 + \|z\|_{L^2(0, T; H^1(\Omega))}^2 + \|z_x\|_{C([0, 1]; L^2(0, T))}^2 \right)^{\frac{1}{2}}.$$

Then, for $z \in Y_T$, by using the semigroup estimates (Bona et al., 2003, Prop 2.1, Prop 2.4, Prop 2.16, Prop 2.17), we have

$$(3.6) \quad \begin{aligned} \|y\|_{Y_T} &= \|\Psi z\|_{Y_T} \leq \|S(t)y_0\|_{Y_T} + \left\| \int_0^t S(t-s)Fz(s)ds \right\|_{Y_T} \\ &\leq c_0(1+T)^{\frac{1}{2}}\|y_0\|_{L^2(\Omega)} + c_1(1+T)^{\frac{1}{2}}\|-\lambda z + az_x(0, \cdot)\|_{L^1(0, T; L^2(\Omega))} \\ &\leq c_0(1+T)^{\frac{1}{2}}\|y_0\|_{L^2(\Omega)} + c_1(1+T)^{\frac{1}{2}}\sqrt{T}(1+\|a\|_{L^2(\Omega)})\|z\|_{Y_T}, \end{aligned}$$

where c_0 and c_1 are positive constants which do not depend on the varying parameters. It follows that Ψ maps Y_T into itself. Now, let $z_1, z_2 \in Y_T$ and $y_1 = \Psi z_1, y_2 = \Psi z_2$. By using similar arguments, we have

$$\|y_1 - y_2\|_{Y_T} = \|\Psi z_1 - \Psi z_2\|_{Y_T} \leq c_1(1+T)^{\frac{1}{2}}\sqrt{T}(1+\|a\|_{L^2(\Omega)})\|z_1 - z_2\|_{Y_T}.$$

Let $T \in (0, T')$ be such that

$$0 < (1 + T)^{\frac{1}{2}} \sqrt{T} < \left(\frac{1}{c_1 (1 + \|a\|_{L^2(\Omega)})} \right).$$

Then, Ψ is a contraction on Y_T and this gives us a unique local solution $y \in Y_T$. Here, the size of T is independent of the size of the initial datum. This will contrast with the corresponding nonlinear model in which the size of T will be related to the size of the initial datum.

Step 2 - Global wellposedness: Let $T_{\max} \leq T'$ be the maximal time of existence for the local solution found in Step 1 in the sense that $y \in Y_T$ for all $T < T_{\max}$. In order to prove that y is global, and deduce that T can be taken as T' , it is enough to show that $\lim_{T \rightarrow T_{\max}^-} \|y\|_{Y_T} < \infty$. This will be proved via multipliers, which will be done only formally, but the calculations can always be justified by a density argument which relies on the regularity result in part (ii) of this proposition. To this end, we multiply (3.1) by y and integrate over Ω to obtain

$$(3.7) \quad \frac{1}{2} \frac{d}{dt} \|y(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} |y_x(0, t)|^2 + \lambda \|y(t)\|_{L^2(\Omega)}^2 = \int_0^1 a(x) y_x(0, t) y(x, t) dx.$$

Using ϵ -Young's inequality with $\epsilon = \frac{1}{4}$, we have

$$(3.8) \quad \frac{1}{2} \frac{d}{dt} \|y(t)\|_{L^2(\Omega)}^2 + \frac{1}{4} |y_x(0, t)|^2 + \lambda \|y(t)\|_{L^2(\Omega)}^2 \leq \|a(x)\|_{L^\infty(\Omega)}^2 \|y(t)\|_{L^2(\Omega)}^2.$$

Integrating the above inequality over $(0, t)$, we get

$$(3.9) \quad \|y(t)\|_{L^2(\Omega)}^2 + \int_0^t |y_x(0, s)|^2 ds \leq 2 \|y_0\|_{L^2(\Omega)}^2 + 4 (\|a(x)\|_{L^\infty(\Omega)}^2 - \lambda) \int_0^t \|y(s)\|_{L^2(\Omega)}^2 ds.$$

Let

$$E_0(t) := \|y(t)\|_{L^2(\Omega)}^2 + \int_0^t |y_x(0, s)|^2 ds.$$

Then, from (3.9), we get

$$E_0(t) \leq 2 \|y_0\|_{L^2(\Omega)}^2 + 4 \left| \|a(x)\|_{L^\infty}^2 - \lambda \right| \int_0^t E_0(s) ds.$$

Now, thanks to the Gronwall's lemma, we have

$$(3.10) \quad E_0(t) = \|y(t)\|_{L^2(\Omega)}^2 + \int_0^t |y_x(0, s)|^2 ds \leq 2 \|y_0\|_{L^2(\Omega)}^2 e^{4 \left| \|a(x)\|_{L^\infty}^2 - \lambda \right| t}, \quad t \in [0, T_{\max}).$$

We in particular deduce that

$$(3.11) \quad \lim_{T \rightarrow T_{\max}^-} \|y\|_{C([0, T]; L^2(\Omega))} \leq \sqrt{2} \|y_0\|_{L^2(\Omega)} e^{2 \left| \|a(x)\|_{L^\infty}^2 - \lambda \right| T_{\max}} < \infty.$$

By using (3.10), we also deduce that

$$(3.12) \quad \lim_{T \rightarrow T_{\max}^-} \|y\|_{L^2(0, T; L^2(\Omega))} \leq \sqrt{2 T_{\max}} \|y_0\|_{L^2(\Omega)} e^{2 \left| \|a(x)\|_{L^\infty}^2 - \lambda \right| T_{\max}}.$$

Secondly, we multiply (3.1) by xy and integrate over $\Omega \times (0, t)$ and get

$$(3.13) \quad \int_0^1 xy^2(x, s)dx + 3 \int_0^t \int_0^1 y_x^2(x, s)dx ds + \lambda \int_0^t \int_0^1 xy^2(x, s)dx ds \\ = \int_0^1 xy_0^2(x)dx + \int_0^t \int_0^1 y^2(x, s)dx ds + \int_0^t \int_0^1 xay_x(0, s)y(x, s)dx ds.$$

From the above identity, it follows that

$$(3.14) \quad \|y_x\|_{L^2(0,t;L^2(\Omega))}^2 \leq \frac{1}{3}\|y_0\|_{L^2(\Omega)}^2 + \left(\frac{1}{2} + \frac{\|a\|_{L^\infty(\Omega)}^2}{18}\right) \int_0^t E_0(s)ds.$$

Combining the above inequality with (3.11), we deduce that

$$(3.15) \quad \lim_{T \rightarrow T_{\max}^-} \|y_x\|_{L^2(0,T;L^2(\Omega))} \\ \leq \frac{1}{\sqrt{3}}\|y_0\|_{L^2(\Omega)} + \left(\frac{1}{\sqrt{2}} + \frac{\|a\|_{L^\infty(\Omega)}}{3\sqrt{2}}\right) \sqrt{2T_{\max}} \left(\|y_0\|_{L^2(\Omega)} e^{2\|a(x)\|_{L^\infty}^2 - \lambda|T_{\max}}\right).$$

Using (3.12) and (3.15), we deduce that

$$(3.16) \quad \lim_{T \rightarrow T_{\max}^-} \|y\|_{L^2(0,T;H^1(\Omega))} \\ \leq \frac{1}{\sqrt{3}}\|y_0\|_{L^2(\Omega)} + \left(1 + \frac{1}{\sqrt{2}} + \frac{\|a\|_{L^\infty(\Omega)}}{3\sqrt{2}}\right) \sqrt{2T_{\max}} \left(\|y_0\|_{L^2(\Omega)} e^{2\|a(x)\|_{L^\infty}^2 - \lambda|T_{\max}}\right) < \infty.$$

Since y is the fixed point in (3.4), we have

$$(3.17) \quad y = S(t)y_0 + \int_0^t S(t-s)Fy(s)ds.$$

Using (Bona et al., 2003, Prop 2.16 and Prop 2.17), we know that the semigroup enjoys the properties

$$(3.18) \quad \sup_{x \in \Omega} \|\partial_x[S(t)y_0](x)\|_{L^2(0,T)} \leq c_2\|y_0\|_{L^2(\Omega)}$$

and

$$(3.19) \quad \sup_{x \in \Omega} \left\| \partial_x \left[\int_0^t S(t-s)Fy(s)ds \right] (x) \right\|_{L^2(0,T)} \leq c_3 \int_0^T \|[Fy](\cdot, t)\|_{L^2(\Omega)} dt$$

for some $c_2, c_3 > 0$. From the definition of Fy we have

$$\|[Fy](\cdot, t)\|_{L^2(\Omega)} \leq \lambda\|y(\cdot, t)\|_{L^2(\Omega)} + \|a\|_{L^2(\Omega)}|y_x(0, t)|.$$

Therefore, by (3.10) and the Cauchy-Schwarz inequality, we have the estimate

$$(3.20) \quad \int_0^T \|[Fy](\cdot, t)\|_{L^2(\Omega)} dt \leq \lambda \int_0^T \sqrt{E_0(t)}dt + \|a\|_{L^2(\Omega)} \sqrt{T} \sqrt{E_0(T)} \\ \leq \left(\lambda T_{\max} + \|a\|_{L^2(\Omega)} \sqrt{T_{\max}}\right) \sqrt{2}\|y_0\|_{L^2(\Omega)} e^{2\|a(x)\|_{L^\infty}^2 - \lambda|T_{\max}}.$$

Now, it follows from (3.18)-(3.20) that

$$\begin{aligned} & \lim_{T \rightarrow T_{\max}^-} \|y_x\|_{C([0,1];L^2(0,T))} \\ & \leq c_2 \|y_0\|_{L^2(\Omega)} + c_3 \left(\lambda T_{\max} + \|a\|_{L^2(\Omega)} \sqrt{T_{\max}} \right) \sqrt{2} \|y_0\|_{L^2(\Omega)} e^{2\|a(x)\|_{L^\infty}^2 - \lambda} T_{\max} < \infty. \end{aligned}$$

Step 3 - Regularity: Regarding the regular solutions, assume that $y_0 \in D(A)$ and consider the following problem:

$$(3.21) \quad \begin{cases} q_t + q_x + q_{xxx} + \lambda q = a(\cdot)q_x(0, \cdot) & \text{in } \Omega \times (0, T), \\ q(0, t) = q(1, t) = q_x(1, t) = 0 & \text{in } (0, T), \\ q(x, 0) = q_0 \equiv -y_0'(x) - y_0'''(x) - \lambda y_0(x) + y_0'(0)a(x) & \text{in } \Omega. \end{cases}$$

Note that $q_0 \in L^2(\Omega)$ and we can solve (3.21) in Y_T as before. Now, we set

$$y(x, t) := y_0(x) + \int_0^t q(x, s) ds.$$

Then,

$$(3.22) \quad \begin{aligned} & y_t(x, t) + y_x(x, t) + y_{xxx}(x, t) + \lambda y(x, t) - a(x)y_x(0, t) \\ & = q(x, t) + y_0'(x) + y_0'''(x) + \lambda y_0 - y_0'(0)a(x) \\ & \quad + \int_0^t (q_x(x, s) + q_{xxx}(x, s) + \lambda q(x, t) - a(x)q_x(0, s)) ds = 0, \end{aligned}$$

and moreover $y(x, 0) = y_0$ and $y(0, t) = y(1, t) = y_x(1, t) = 0$. Therefore, y solves (3.1). Writing

$$y_{xxx}(x, t) = -q(x, t) - y_x(x, t) - \lambda y(x, t) + a(x)y_x(0, t)$$

and taking $L^2(\Omega)$ norms of both sides we get

$$\|\partial_x^3 y(t)\|_{L^2(\Omega)} \leq \|q(t)\|_{L^2(\Omega)} + \|\partial_x y(t)\|_{L^2(\Omega)} + \lambda \|y(t)\|_{L^2(\Omega)} + |y_x(0, t)| \|a\|_{L^2(\Omega)}.$$

Recall that we have the Gargliardo-Nirenberg inequalities

$$\|\partial_x y(t)\|_{L^2(\Omega)} \lesssim \|y\|_{L^2(\Omega)}^{\frac{2}{3}} \|\partial_x^3 y\|_{L^2(\Omega)}^{\frac{1}{3}},$$

$$\|\partial_x^2 y(t)\|_{L^2(\Omega)} \lesssim \|y\|_{L^2(\Omega)}^{\frac{1}{3}} \|\partial_x^3 y\|_{L^2(\Omega)}^{\frac{2}{3}},$$

and the trace inequality (remember that $y_x(1, t) = 0$)

$$|y_x(0, t)| \leq \|\partial_x^2 y\|_{L^2(\Omega)}.$$

Using these estimates, we get

$$\|\partial_x^3 y(t)\|_{L^2(\Omega)} \lesssim \|q(t)\|_{L^2(\Omega)} + \|y(t)\|_{L^2(\Omega)}.$$

By taking the sup norm with respect to the temporal variable, we deduce that

$$y \in C([0, T]; H^3(\Omega)).$$

Similarly, writing out

$$\partial_x^4 y(x, t) = -q_x(x, t) - y_{xx}(x, t) - \lambda y_x(x, t) + a'(x)y_x(0, t),$$

using the Gagliardo-Nirenberg and trace inequality, we get

$$\|\partial_x^4 y(t)\|_{L^2(\Omega)} \lesssim \|q_x(t)\|_{L^2(\Omega)} + \|y_x(t)\|_{L^2(\Omega)}.$$

Taking $L^2(0, T)$ norms of both sides we deduce that $y \in L^2(0, T; H^4(\Omega))$. \square

Global well-posedness of the linearized model (2.17) now follows from the Proposition 3.2 that we have just proved.

Remark 3.23. *One can interpolate between part (i) and part (ii) of the above proposition with respect to the smoothness of initial data and get the corresponding well-posedness and regularity result in fractional spaces. For example, let $y_0 \in H^s(\Omega)$ ($s \in [0, 3]$) so that it satisfies the compatibility conditions $y_0(0) = y_0(1) = 0$ if $s \in [0, 3/2]$ and the compatibility conditions $y_0(0) = y_0(1) = y_0'(1) = 0$ if $s \in (3/2, 3]$. Then, with say a sufficiently smooth, one has*

$$y \in Y_T^s := \{\psi \in C([0, T]; H^s(\Omega)) \cap L^2(0, T; H^{s+1}(\Omega)) \mid \psi_x \in C([0, 1]; L^2(0, T))\}.$$

The arguments in Step 3 of the proof of above proposition can be easily extended to the nonhomogeneous equation

$$y_t + y_x + y_{xxx} + \lambda y = a(x)y_x(0, \cdot) + f.$$

One can first study this equation with $s = 0$, $f \in L^1(0, T; L^2(\Omega))$, secondly with $s = 3$, $f \in W^{1,1}(0, T; L^2(\Omega))$. Then, by interpolation, for $s \in (0, 3)$, one can get $y \in Y_T^s$ if $f \in W^{s/3, 1}(0, T; L^2(\Omega))$. Moreover, the following estimates are true:

$$(3.24) \quad \|y\|_{Y_T^s} \lesssim \|y_0\|_{H^s(\Omega)} + \|f\|_{W^{s/3, 1}(0, T; L^2(\Omega))},$$

and for $s = 3$,

$$(3.25) \quad \|y_t\|_{Y_T} \lesssim \|y_0\|_{H^3(\Omega)} + \|f\|_{W^{1,1}(0, T; L^2(\Omega))}.$$

3.2. Nonlinear model. Consider the following nonlinear KdV equation with homogeneous boundary conditions.

$$(3.26) \quad \begin{cases} y_t + y_x + y_{xxx} + \lambda y = a(x)y_x(0, \cdot) - (y + v)(y_x + v_x) & \text{in } \Omega \times \mathbb{R}_+, \\ y(0, t) = y(1, t) = y_x(1, t) = 0 & \text{in } \mathbb{R}_+, \\ y(x, 0) = y_0 \in L^2(\Omega) & \text{in } \Omega, \end{cases}$$

where $v = \Phi(y)$ with Φ being the linear operator defined in Section 2.1 in the proof of Lemma 2.22.

Proposition 3.27. *i) Let $T' > 0$ be arbitrary and $y_0, a \in L^2(\Omega)$. Then, there exists $T \in (0, T')$ depending on the size of y_0 such that (3.26) has a unique local solution*

$$y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

satisfying also

$$y_x \in C([0, 1]; L^2(0, T)).$$

Moreover, if $a \in L^\infty(\Omega)$ and $\|y_0\|_{L^2(\Omega)}$ is sufficiently small, then y extends as a global solution. In other words, T can be taken as T' .

ii) Let $a \in H^1(\Omega)$ and $y_0 \in H^3(\Omega)$ satisfying the computability conditions

$$y_0(0) = y_0(1) = y_0'(1) = 0.$$

Then, the local solution in part (i) enjoys the extra regularity

$$y \in C([0, T]; H^3(\Omega)) \cap L^2(0, T; H^4(\Omega)).$$

Proof. Step 1 - Local wellposedness: At first, we set a nonlinear operator Y as follows:

$$(3.28) \quad y = [Yz](t) := S(t)y_0 + \int_0^t S(t-s)Fz(s)ds,$$

where

$$Fz := -\lambda z + a(\cdot)z_x(0, \cdot) - (z + v)(z_x + v_x)$$

with $v = \Phi(z)$. Here, we consider Y on a set given by

$$S_{T,r} := \{z \in Y_T, \|z\|_{Y_T} \leq r\},$$

where Y_T is as in (3.5). The parameters $T, r > 0$ will be determined later. $S_{T,r}$ is a complete metric subspace of Y_T with respect to the metric induced by the norm of Y_T . Since $v = \Phi z$, due to (2.24) we have

$$(3.29) \quad \|v\|_{C([0,T];L^2(\Omega))} \leq C_0 \|z(t)\|_{C([0,T];L^2(\Omega))}.$$

Similarly, using (2.27) we deduce

$$(3.30) \quad \|v\|_{L^2(0,T);H^1(\Omega)} \leq C_1 \|z(t)\|_{L^2(0,T);L^2(\Omega)}.$$

Finally,

$$(3.31) \quad \|v_x(x)\|_{L^2(0,T)}^2 = \int_0^T \left| \int_0^x \tilde{k}_x(x,y)z(y,t)dy \right|^2 dt \\ \leq \left(\int_0^1 |\tilde{k}_x(x,y)|^2 dy \right) \|z\|_{L^2(0,T);L^2(\Omega)}^2,$$

from which it follows that

$$(3.32) \quad \sup_{x \in (0,1)} \|v_x(x)\|_{L^2(0,T)} \leq \|z\|_{L^2(0,T);L^2(\Omega)} \sup_{x \in (0,1)} \left(\int_0^1 |\tilde{k}_x(x,y)|^2 dy \right)^{\frac{1}{2}}.$$

Combining (3.29), (3.30), and (3.32), we have

$$(3.33) \quad \|v\|_{Y_T} \leq c_{\tilde{k}} \|z\|_{Y_T},$$

where $c_{\tilde{k}} > 0$ is a constant which only depends on various finite norms of \tilde{k} . Taking the Y_T norm of both sides of (3.28) and using the same semigroup estimates on Y_T , we

obtain

$$\begin{aligned}
(3.34) \quad \|Yz\|_{Y_T} &\leq c_0 \|y_0\|_{Y_T} + c_1 \int_0^T \| [Fz](\cdot, s) \|_{L^2(\Omega)} ds \\
&\leq c_0 \|y_0\|_{Y_T} + c_1 \int_0^T \| a(\cdot)z_x(0, s) - \lambda z - (z + v)(z_x + v_x) \|_{L^2(\Omega)} ds \\
&\leq c_0 \|y_0\|_{Y_T} \\
&+ c_1 \int_0^T \left[\| a(\cdot)z_x(0, s) - \lambda z \|_{L^2(\Omega)} + \| zz_x \|_{L^2(\Omega)} + \| zv_x \|_{L^2(\Omega)} + \| vz_x \|_{L^2(\Omega)} + \| vv_x \|_{L^2(\Omega)} \right] ds \\
&\leq c_0 \|y_0\|_{Y_T} \\
&+ c_1 \left[(1 + T)^{\frac{1}{2}} \sqrt{T} (1 + \|a\|_{L^2(\Omega)}) \|z\|_{Y_T} + (T^{\frac{1}{2}} + T^{\frac{1}{3}}) \left(\|z\|_{Y_T}^2 + 2\|z\|_{Y_T} \|v\|_{Y_T} + \|v\|_{Y_T}^2 \right) \right] \\
&\leq c_0 \|y_0\|_{Y_T} + c_1 \left[(1 + T)^{\frac{1}{2}} \sqrt{T} (1 + \|a\|_{L^2(\Omega)}) + (1 + 3c_{\tilde{k}})(T^{\frac{1}{2}} + T^{\frac{1}{3}}) \|z\|_{Y_T} \right] \|z\|_{Y_T},
\end{aligned}$$

where the fourth inequality follows from (Bona et al., 2003, Lemma 3.1). Let us set $r = 2c_0 \|y_0\|_{Y_T}$, and choose $T > 0$ to be small enough that

$$c_1 \left[(1 + T)^{\frac{1}{2}} \sqrt{T} (1 + \|a\|_{L^2(\Omega)}) + (1 + 3c_{\tilde{k}})(T^{\frac{1}{2}} + T^{\frac{1}{3}}) r \right] \leq \frac{1}{2}.$$

With such choice of (r, T) , we get $\|Yz\|_{Y_T} \leq r$ for all $z \in S_{T,r}$. Therefore, Y is a map from $S_{T,r}$ into $S_{T,r}$.

Now, we claim that Y is indeed a contraction on $S_{T,r}$ if T is sufficiently small. In order to see this, let $z, z' \in S_{T,r}$. Then, similar to the (3.34), we have

$$\begin{aligned}
(3.35) \quad \|Yz - Yz'\|_{Y_T} &\leq c_1 \int_0^T \| [Fz - Fz'](\cdot, s) \|_{L^2(\Omega)} ds \\
&\leq c_1 \int_0^T \| a(\cdot)(z_x(0, s) - z'_x(0, s)) - \lambda(z - z') \|_{L^2(\Omega)} ds \\
&+ c_1 \int_0^T \left[\| zz_x - z'z'_x \|_{L^2(\Omega)} + \| zv_x - z'v'_x \|_{L^2(\Omega)} + \| vz_x - v'z'_x \|_{L^2(\Omega)} + \| vv_x - v'v'_x \|_{L^2(\Omega)} \right] ds \\
&\leq c_1 (1 + T)^{\frac{1}{2}} \sqrt{T} (1 + \|a\|_{L^2(\Omega)}) \|z - z'\|_{Y_T} + c_1 (T^{\frac{1}{2}} + T^{\frac{1}{3}}) (\|z\|_{Y_T} + \|z'\|_{Y_T}) \|z - z'\|_{Y_T} \\
&+ c_1 (T^{\frac{1}{2}} + T^{\frac{1}{3}}) (\|z'\|_{Y_T} \|v - v'\|_{Y_T} + \|v\|_{Y_T} \|z - z'\|_{Y_T}) + c_1 (T^{\frac{1}{2}} + T^{\frac{1}{3}}) (\|z\|_{Y_T} \|v - v'\|_{Y_T} + \|v\|_{Y_T} \|z - z'\|_{Y_T}) \\
&\quad + c_1 (T^{\frac{1}{2}} + T^{\frac{1}{3}}) (\|v\|_{Y_T} + \|v'\|_{Y_T}) \|v - v'\|_{Y_T}.
\end{aligned}$$

Now, using (3.33), for the same r as before, but choosing T smaller if necessary, we obtain

$$\|Yz - Yz'\|_{Y_T} \leq \rho \|z - z'\|_{Y_T}$$

for some $\rho \in (0, 1)$. Then, by the Banach contraction theorem, we get the existence and uniqueness of a local solution in $S_{T,r}$.

Step 2 - Regularity: Let $y_0 \in D(A)$. We define the closed space

$$B_{T,r} := \{(\psi, \varphi) \in Y_T^3 \times Y_T \mid \psi \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \varphi = \psi_t, \|\psi\|_{Y_T^3} + \|\varphi\|_{Y_T} \leq r\}.$$

Now, given $(z, \tilde{z}) \in B_{T,r}$ let q be a solution of

$$(3.36) \quad \begin{cases} q_t + q_x + q_{xxx} + \lambda q = a(x)q_x(0, \cdot) - (\tilde{z} + \tilde{v})(z_x + v_x) - (z + v)(\tilde{z}_x + \tilde{v}_x) & \text{in } \Omega \times (0, T), \\ q(0, t) = q(1, t) = q_x(1, t) = 0 & \text{in } (0, T), \\ q(x, 0) = q_0 := -y'_0 - y'''_0 - \lambda y_0 + a(x)y'_0(0) - (y_0 + v_0)(y'_0 + v'_0) & \text{in } \Omega, \end{cases}$$

where $v = \Phi(z)$, $v_0 = \Phi(y_0)$, $\tilde{v} = \Phi(\tilde{z})$. Set $y = y_0 + \int_0^t q ds$. Then, $y_t = q$ and y solves

$$(3.37) \quad \begin{cases} y_t + y_x + y_{xxx} + \lambda y = a(x)y_x(0, \cdot) - (z + v)(z_x + v_x) & \text{in } \Omega \times (0, T), \\ y(0, t) = y(1, t) = y_x(1, t) = 0 & \text{in } (0, T), \\ y(x, 0) = y_0 & \text{in } \Omega. \end{cases}$$

We set an operator

$$\Theta : (z, \tilde{z}) \mapsto (y, q)$$

associated to the system of equations given by (3.36)-(3.37). One can show that for suitable r and small T , the operator Θ maps $B_{T,r}$ onto itself in a contractive manner. This can be done by obtaining the same type of estimates given in Step 1 for both the solution of (3.36) and (3.37). Therefore, it has a unique fixed point whose first component is the regular local solution we are looking for.

Step 3 - Global solutions: Global wellposedness in Y_T with small initial datum follows directly from the stabilization estimate proved in Section 2.2. \square

Global well-posedness of the nonlinear modified target system (2.37) now follows from the Proposition 3.27 that we have just proved.

4. NUMERICAL SIMULATIONS

We modify the finite difference scheme given in (Pazoto et al., 2010) to fit it into the present situation, where we have first order trace terms in the main equations of the target systems and moreover the original plant involves inhomogeneous boundary inputs of feedback type. We numerically solve the KdV equation both in the controlled and uncontrolled cases. We are able to verify our main result also numerically. At first, we simulate an uncontrolled solution of the KdV equation and then we simulate the controlled solution. From our simulations, one can see that the boundary controllers constructed by using a pseudo-kernel effectively stabilizes the solutions with a suitable choice of λ . The calculations are performed in Wolfram Mathematica[®] 11.

For simplicity, we consider only the linearised problem. The nonlinear problem can be treated in a similar way by including an additional fixed point argument to the algorithm we describe here. We adapt to the notation given in (Pazoto et al., 2010). To this end, we set the discrete space

$$X_J := \{\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_J) \in \mathbb{R}^{J+1} \mid \tilde{w}_0 = \tilde{w}_{J-1} = \tilde{w}_J = 0\},$$

and the difference operators $(D^+ \tilde{w})_j := \frac{\tilde{w}_{j+1} - \tilde{w}_j}{\delta x}$, $(D^- \tilde{w})_j := \frac{\tilde{w}_j - \tilde{w}_{j-1}}{\delta x}$ for $j = 1, \dots, J-1$, and $D = \frac{1}{2}(D^+ + D^-)$. We will call δx and δt to be the space and time

steps, for $j = 0, \dots, J$, and $n = 0, 1, \dots, N$, respectively. Using this notation, the numerical approximation of the linearised target system (2.17) takes the form

$$(4.1) \quad \frac{\tilde{w}_j^{n+1} - \tilde{w}_j^n}{\delta t} + (\mathcal{A}\tilde{w}^{n+1})_j + \lambda\tilde{w}_j^{n+1} = \tilde{k}_y(x_j, 0) \frac{\tilde{w}_1^n}{\delta x}, \quad j = 1, \dots, J-1$$

$$(4.2) \quad \tilde{w}_0 = \tilde{w}_{J-1} = \tilde{w}_J = 0,$$

$$(4.3) \quad \tilde{w}_0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{w}_0(x) dx, \quad j = 1, \dots, J-1,$$

where $x_{j\mp\frac{1}{2}} = (j\mp\frac{1}{2})\delta x$, $x_j = j\delta x$. The $(J-1) \times (J-1)$ matrix \mathcal{A} approximates $\tilde{w}_x + \tilde{w}_{xxx}$ and it is defined by $\mathcal{A} := D^+D^+D^- + D$. Let us set $\mathcal{C} := (1 + \delta t\lambda)I + \delta t\mathcal{A}$. Then, from the main equation, we obtain

$$\tilde{w}_j^{n+1} = \mathcal{C}^{-1} \left(\tilde{w}_j^n + \frac{\delta t}{\delta x} \tilde{k}_y(x_j, 0) \tilde{w}_1^n \right)$$

for $j = 1, \dots, J-1$.

In order to approximate the solution of the original plant (1.1) with feedback controllers, we use the succession idea in the proof of Lemma 2.22. Note that given \tilde{w} , v is the fixed point of the equation $v = K(\tilde{w} + v)$. For numerical purposes, let m denote the number of iterations in the succession and set $v^0 = \mathcal{K}\tilde{w}$, $v^k := \mathcal{K}(\tilde{w} + v^{k-1})$ for $1 \leq k \leq m$, where \mathcal{K} is the numerical approximation of the integral in the definition of K . Then, v^m is an approximation of $v = \Phi(\tilde{w})$, and one gets an approximation of the original plant by setting $u(x_j, t_n) := \tilde{w}(x_j, t_n) + v^m(x_j, t_n)$.

On a domain of critical length, one can find time independent solutions as we have mentioned in the introduction. Figure 9 below shows such a solution on a domain of length $L = 2\pi$ whose L^2 -norm is preserved in time.

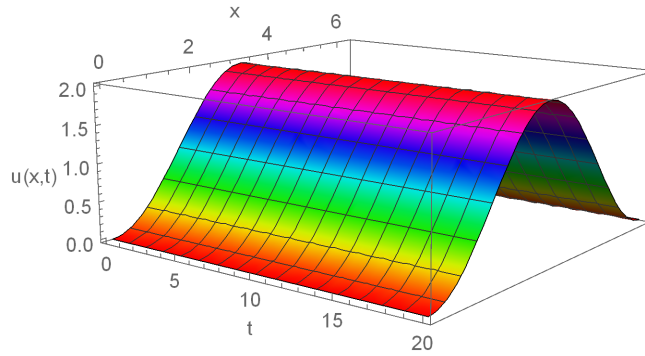


FIGURE 9. Uncontrolled solution with initial datum $u_0 = 1 - \cos(x)$ on a domain of length 2π .

If one applies the boundary controllers constructed with the same initial profile that the uncontrolled solution has in Figure 9, then the new solution will decay to zero as we illustrate in Figure 10.

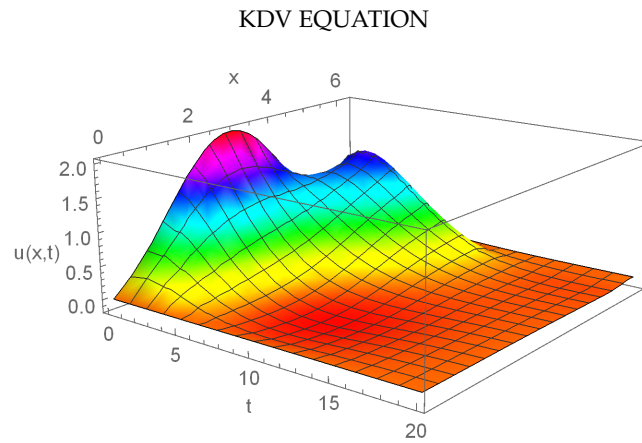


FIGURE 10. Controlled solution with initial datum $u_0 = 1 - \cos(x)$, $\lambda = 0.03$, with a controller using the pseudo-kernel \tilde{k} on a domain of length 2π .

Figure 11 shows the controller behavior on the Dirichlet boundary condition at the right endpoint. As one can see, less control is needed as the wave gets suppressed.

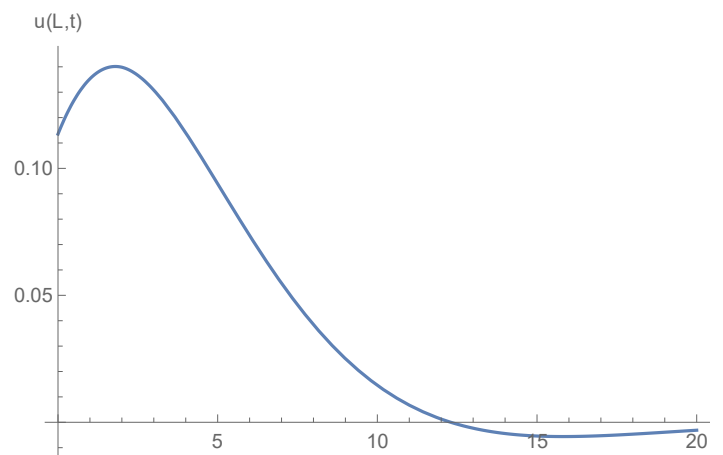


FIGURE 11. Dirichlet controller at the right end point ($\lambda = 0.03$)

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