Feedback Control of Dynamic Systems

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Contents

• Introduction to Control Theory
• Modeling Physical Systems
• Laplace Transforms
First Control Application: Watt’s Speed Governor (1769)
Achievements

• Watt was a practical engineer. He did not make theoretical analysis. But he showed that the speed oscillates under certain conditions (instability)
• Maxwell (1868) developed a model using dif. Eq. and found stability criteria (roots of characteristic equation must have negative real parts)
• Stability of non-linear systems studied by Lyapunov (1893)
• Mathematical framework established by Laplace and Fourier
• Nyquist (1932), Bode and Nichols (1945) used frequency domain methods based on complex variables to determine system stability.
• Evans (1948) found a graphical method: Root-Locus
• Kalman (1961) announced optimal control strategies.
• In 1980s, $H_\infty$ control was used to overcome dynamic uncertainties (robust control methods)
• In 1990s, intelligent control methods studied (Fuzzy-logic controllers)
Open-loop Control

For example, think of the temperature control of a room.

Advantage: Simple
Disadvantage: Disturbance cannot be taken into account
Closed-loop Control

Diagram showing the components of a closed-loop control system:
- Desired Value
- Summing Point
- Error Signal
- Controller
- Control Signal
- Plant
- Measured Value
- Sensor
- Feedback Path
- Output Value
Aircraft Fluid-Level Control
Room Temperature Control System

Desired Temperature

Potentiometer

Control Signal

Gas Solenoid Valve

Gas Flow-rate

Outside Temperature

Heat Loss

Actual Room Temperature

Heat Input

Thermometer

Inside Temperature

Desired Temperature

Potentiometer

Error Signal

Controller

Control Signal

Gas Solenoid Valve

Gas Flow-rate

Heat Loss

Insulation

Room

Actual Temperature

(°C)

(V)

(°C)

(V)

(°C)
Ship Autopilot Control System
Modeling Physical Systems
A simple model for a motor vehicle

\[ \Theta(t) : \text{accelerator pedal angle} \quad u(t) : \text{forward speed} \]

\[ u(t) = a \cdot \Theta(t) \]

\( D: \text{aerodynamic drag} \)
\( F: \text{wheel traction force} \)
\( T: \text{engine torque} \)

\[ T = b \cdot \Theta(t) \]
\[ F = c \cdot T \]
\[ D = d \cdot u(t) \]

\[ D \text{ must equal to } F \quad \Rightarrow \quad D = F \]
\[ d\ u(t) = c \cdot T \]

\[ u(t) = \left( \frac{cb}{d} \right) \cdot \Theta(t) \]

\( \Theta(t) : \text{degrees} \quad a = \frac{b \cdot c}{d} \)

This model is not realistic since we know that it takes time to build up to the new forward speed. We use diff. eq.

\[ e \cdot \frac{du}{dt} + fu = g \cdot \Theta(t) \]

acceleration, when it travels at constant velocity, becomes zero

\[ so \quad f\ u(t) = g \cdot \Theta(t) \quad \Rightarrow \quad u(t) = \left( \frac{g}{f} \right) \cdot \Theta(t) \quad (a = \frac{g}{f}) \quad \text{the same equation again!} \]

Using diff. eq. to represent systems

1st order system: \( a \cdot \dot{y} + b \cdot y = c \cdot x(t) \)

2nd order system: \( a \cdot \ddot{y} + b \cdot \dot{y} + c \cdot y = e \cdot x(t) \)

3rd order system: \( a \cdot \dddot{y} + b \cdot \ddot{y} + c \cdot \dot{y} + e \cdot y = f \cdot x(t) \)

\( a, b, c, e \ldots \text{constant coefficients}. \text{ These are linear dif. eq.s.} \)
Mathematical Models for Electrical Systems

\[ \frac{R}{M} i(t) + \epsilon_0 = R_i + i(t) \] (Resistance)

\[ L + \epsilon_0 = L \frac{d}{dt} i(t) \] (Inductance)

\[ C + \epsilon_0 = C \frac{d}{dt} i(t) \] (Capacitance)

\[ \epsilon_1(t) - \epsilon_2(t) = R \cdot i(t) \]

\[ \epsilon_2(t) = -\frac{1}{C} \int i(t) \, dt \quad \text{or} \quad i(t) = -\frac{C}{\epsilon_2(t)} \frac{d\epsilon_2(t)}{dt} \]

\[ \epsilon_1(t) - \epsilon_2(t) = RC \frac{d\epsilon_2(t)}{dt} \]

\[ RC \frac{d\epsilon_2(t)}{dt} + \epsilon_2(t) = \epsilon_1(t) \] (1st order diff. eq.)

\[ \epsilon_1(t) - \epsilon_2(t) = R \cdot i(t) + L \frac{d}{dt} i(t) \]

\[ \epsilon_2(t) = -\frac{1}{C} \int i(t) \, dt \quad \text{or} \quad i(t) = -\frac{C}{\epsilon_2(t)} \frac{d\epsilon_2(t)}{dt} \]

\[ \epsilon_1(t) - \epsilon_2(t) = RC \frac{d\epsilon_2(t)}{dt} + L \frac{d}{dt} \left( c \frac{d\epsilon_2(t)}{dt} \right) \]

\[ LC \frac{d^2\epsilon_2(t)}{dt^2} + RC \frac{d\epsilon_2(t)}{dt} + \epsilon_2(t) = \epsilon_1(t) \] (2nd order diff. eq.)

\[ i_1, i_2, R_1, R_2, C_1, C_2 \]

\[ \epsilon_1 - \epsilon_2 = R_1 (i_1 + i_2) \]

\[ C_1 \frac{d\epsilon_2}{dt} = i_2 \]

\[ \epsilon_2 - \epsilon_2 = R_2 i_2 \]

\[ i_2 = C_2 \frac{d\epsilon_2}{dt} \]

\[ R, R_2, C_1, C_2 \frac{d^2\epsilon_2}{dt^2} + (R_1 C_1 + R_1 C_2 + R_2 C_2) \frac{d\epsilon_2}{dt} + \epsilon_2 = \epsilon_1 \]
Heat flow by conduction is given by Fourier's Law:
\[ Q_T = KA \left( \theta_1 - \theta_2 \right) \]

- \( K \): Thermal conductivity (W/mK)
- \( Q_T \): Heat flow (J/s = W)

If \( R_T = \frac{\ell}{KA} \) (Thermal resistance), then
\[ \theta_1(t) \theta_2(t) = R_T Q_T(t) \]

**Thermal capacitance:**

The heat stored by a body is
\[ H(t) = m C_p \frac{\theta(t)}{C_T} \]

- \( H \): heat (J)
- \( m \): mass (kg)
- \( C_p \): specific heat at constant pressure (J/kgK)
- \( \theta \): temperature (K)

To obtain heat flow \( Q_T(t) \) = \[ \frac{d}{dt} H(t) = C_T \frac{d\theta(t)}{dt} \]

**Wall:**

Heat source \( \theta_1(t) \)

Heat sink \( \theta_2(t) \)

Heat flow is given by
\[ Q_T = \frac{\theta_1 - \theta_2}{R_T} \quad Q_T = C_T \frac{d\theta_2}{dt} \]

\[ \frac{\theta_1 - \theta_2}{R_T} = C_T \frac{d\theta_2}{dt} \Rightarrow \]

\[ R_T C_T \frac{d\theta_2}{dt} + \theta_2 = \theta_1(t) \]
Stiffness, damping and mass:

An elastic element is assumed to produce an extension proportional to the force applied to it.

**Force & Extension**

\[ F(t) = K(x(t) - x_0(t)) \]

Linear elastic elements (Translational spring)

**Torque & Twist**

\[ T(t) = K(\theta(t) - \theta_0(t)) \]

Rotational spring

A damping element is assumed to produce a velocity proportional to the force applied to it.

**Force & Velocity**

\[ F(t) = C \delta(t) = C \frac{dx_0}{dt} \]

Translational damper

**Torque & Angular velocity**

\[ T(t) = C \omega(t) = C \frac{d\theta_0}{dt} \]

Rotational damper

The force to accelerate a body is the product of its mass and acceleration (Newton's second law)

**Force & Acceleration**

\[ F(t) = ma(t) = m \frac{dv}{dt} = m \frac{d^2x}{dt^2} \]

Free body diagram

\[ \sum F_x = ma \]

\[ K(x(t) - x_0(t)) = C \frac{dv(t)}{dt} = m \frac{d^2x_0}{dt^2} \]

\[ m \frac{d^2x_0}{dt^2} + C \frac{dx_0}{dt} + Kx_0 = Kx_1(t) \]

(2nd order)
If the mass is neglected, then:
\[ C \frac{dx_0}{dt} + K x_0 = K x_1(t) \]
\Rightarrow \text{it becomes a 1st order system}

Example 1

A flywheel of moment of inertia \( I \) sits in bearings that produce a frictional moment of \( C \) times the angular velocity \( \omega(t) \) of the shaft. Find the diff. eq. relating the applied torque \( T(t) \) and the angular velocity \( \omega(t) \).

\[ z M = I \alpha \]
\[ T(t) = C \omega = I \frac{d\omega}{dt} \Rightarrow I \frac{d\omega}{dt} + C \omega = T(t) \text{ or } I \omega + C \omega = T(t) \]

Example 2

Gearbox is driven by a motor that develops a torque \( T_m(t) \). It has a gear reduction ratio of \( n \) and the moments of inertia on the motor and output shafts are \( I_m \) and \( I_o \) and the respective damping coefficients \( C_m \) and \( C_o \). Find diff. eq. modeling this system.

\[ X(t) = \text{gear tooth reduction force} \]

Gearbox parameters:
- \( T_m = 5 \times 10^{-6} \text{ kNm} \)
- \( I_o = 0.01 \text{ kg m}^2 \)
- \( C_m = 60 \times 10^{-6} \text{ Nm s/rad} \)
- \( C_o = 0.15 \text{ Nm s/rad} \)
- \( n = 50 : 1 \)
Cont'd:

Motor shaft:

\[ I M = I_m \frac{d^2 \Theta_m}{dt^2} \]

\[ T_m(t) - C_m \frac{d \Theta_m}{dt} - a \chi(t) = I_m \frac{d^2 \Theta_m}{dt^2} \Rightarrow \chi(t) = \frac{1}{a} \left( T_m(t) - I_m \frac{d^2 \Theta_m}{dt^2} - C_m \frac{d \Theta_m}{dt} \right) \]

Output shaft:

\[ I_d = I_o \frac{d^2 \Theta_o}{dt^2} \]

\[ b \chi(t) - C_o \frac{d \Theta_o}{dt} = I_o \frac{d^2 \Theta_o}{dt^2} \Rightarrow \chi(t) = \frac{1}{b} \left( I_o \frac{d^2 \Theta_o}{dt^2} + C_o \frac{d \Theta_o}{dt} \right) \]

Combining two equations:

\[ \frac{b}{a} \left( T_m(t) - I_m \frac{d^2 \Theta_m}{dt^2} - C_m \frac{d \Theta_m}{dt} \right) = \left( I_o \frac{d^2 \Theta_o}{dt^2} + C_o \frac{d \Theta_o}{dt} \right) \]

Kinematic relationships:

\[ \frac{b}{a} = n, \quad \Theta_m(t) = n \Theta_o(t), \quad \frac{d \Theta_m}{dt} = n \frac{d \Theta_o}{dt}, \quad \frac{d^2 \Theta_m}{dt} = n \frac{d^2 \Theta_o}{dt} \]

Hence:

\[ n \left( T_m(t) - n I_m \frac{d^2 \Theta_o}{dt} - n C_m \frac{d \Theta_o}{dt} \right) = \left( I_o \frac{d^2 \Theta_o}{dt^2} + C_o \frac{d \Theta_o}{dt} \right) \]

\[ (I_o + n^2 I_m) \frac{d^2 \Theta_o}{dt^2} + (C_o + n^2 C_m) \frac{d \Theta_o}{dt} = n T_m(t) \]

\[ 0.0225 \text{ kgm}^2 \quad 0.3 \text{ Nms/rod} \]

\[ 0.0225 \frac{d^2 \Theta_o}{dt^2} + 0.3 \frac{d \Theta_o}{dt} = 50 T_m(t) \]