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Hybrid Dimensional Synthesis of Planar Mechanisms for the Combination of Finite Positions and Path-Points

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Abstract
The combination of the established synthesis methods for finite positions and path-points leads to a hybrid dimensional synthesis method that is presented within this paper. This hybrid method combines the advantages of the established synthesis methods so that an adjustable definition of a synthesis task as well as a high performance of the algorithm can be ensured. To realize this combination, the established algorithms have to be modified as shown within this paper. Depending on the synthesis task and its degree of freedom, a solution can be found analytically or numerically. Both of these algorithms as well as the algorithm for the calculation of the synthesis degree of freedom is treated in this contribution. The shown approaches are validated by two examples.

Keywords: Dimensional Synthesis, Planar Mechanisms, Finite Position, Path-Point

1 Introduction

The process of mechanism synthesis can be divided into the two main tasks: structural synthesis and dimensional synthesis [1]. Based on several boundary conditions, the type synthesis process determines the most suitable structure of a mechanism. That includes the determination of the number and kinds of joints and links. The task of the dimensional synthesis is to identify the kinematic parameters of a chosen structure. So, based on a specific motion task or other specifications, the dimension of each link can be determined.

Both, the structural and the dimensional synthesis highly depend on the definition of the synthesis task. A synthesis task of a guidance mechanism can consist of finite positions or path-points [2]. For a linkage that is based only on revolute or prismatic joints the number of finite positions or path-points is limited. Such a four-bar-mechanism for instance can be synthesized with a maximum number of five finite positions. To determine the maximum number of synthesis task elements that can be realized by a specific mechanism, the value of the synthesis task (sDOF) can be calculated by [2]. Mechanisms can fulfill a synthesis task if their value is greater or equals to the value of this task. A four-bar mechanism with four revolute joints can theoretically fulfill up to five finite positions or nine path-points. This number of possible positions or points decreases if specific joint locations are prescribed.

Nowadays, dimensional synthesis methods mostly focus on either the specification of finite positions or the definition of path-points. Only few, such as [3], developed a method to combine both specifications. The main disadvantage of the developed algorithm in [3] is, that it does not allow the specification of joint locations. A hybrid dimensional synthesis, that allows the definition of finite positions, path-points as well as the location of joints combines the advantages of both established synthesis methods and allows a more adjustable but still robust way to design mechanisms. Some tasks in application, such as pick and place tasks, only require a limited number of finite positions [3–5]. For those tasks a hybrid dimensional synthesis gives the possibility to add additional specifications such as joint locations instead of arbitrary angles of a finite position.

2 Established Dimensional Synthesis Methods

The two most established dimensional synthesis methods are the finite position synthesis based on Burmester [6] and the definition of path-points.

2.1 Finite Position Synthesis

In 1888 Burmester published his approach to synthesize four-bar linkages for the maximum number of five precision positions [6]. Ever since, this approach has been modified and used for the dimensional synthesis of dyads and linkages [7–13].

The basic concept of this approach is the determination of the center-point curve and the corresponding circle-point curve based on the relative displacement poles [9]. Fig. 1 shows an example of these curves for four finite positions. Every point on the center-point curve k_M belongs to an RR chain that reaches all four positions and hence can be used as a stationary revolute joint of a four-bar mechanism. Every center-point has a corresponding circle-point on the circle-point curve k_{K1} which defines the location of the non-stationary revolute joint of the RR chain.

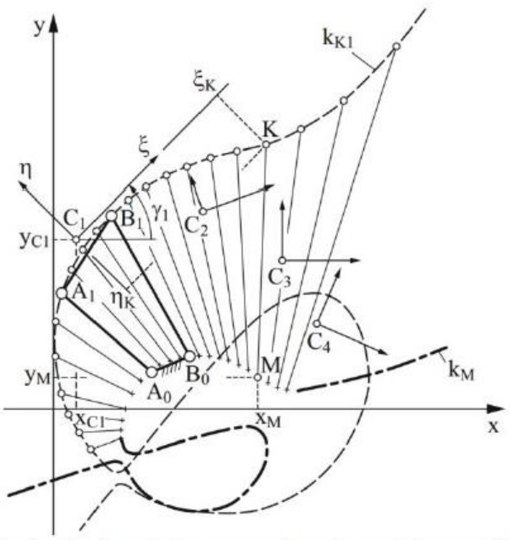

Fig. 1: Circle-point curve and center-point curve [2]

To solve the synthesis task for five finite positions algebraically Dittrich, Braune et al. developed in [7; 14] the nonlinear system of equations (1). This system of equation formulates the relation between the coordinates of the center-points (x_M, y_M) and the coordinates of the circle-point (ξ_K, η_K) in relation to the given finite positions. The coefficients A_j - G_j simply depend on these finite positions (2).

$$A_j + B_j\xi_K + C_j\eta_K + D_j x_M + E_j y_M + F_j(x_M\xi_K + y_M\eta_K) + G_j(y_M\xi_K - x_M\eta_K) = 0 \qquad j = 1,2,3 \tag{1}$$

$$A_j \dots G_j = f(x_{Ci}, x_{Ci+1}, y_{Ci}, y_{Ci+1}, \gamma_i, \gamma_{i+1}) \tag{2}$$

Since this approach leads to a cubic equation shown in [7], the solving algorithm is performant. The problem of this approach is that the synthesis task has to consist of finite positions. So even if a specific angle γ_i is not desired, it has to be defined to use the algorithm based on the Burmester theorem. That limits the solution space and reduces the number of additional boundary conditions.

2.2 Synthesis of Path-Points

The approach of the dimensional synthesis based on the definition of points of a coupler curve provides a relatively adjustable way of synthesis task definition. Here a four-bar linkage can be synthesized by specifying up to nine points of the coupler curve [11; 15; 16]. The unknown locations of the revolute joints can be calculated with the parameters shown in Fig. 2.

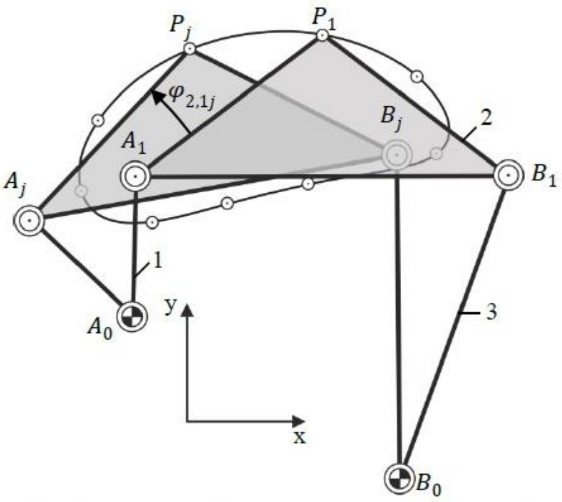

Fig. 2: Four-bar mechanism for the path-point synthesis

This approach is based on the motion of joint A respectively joint B in reference to the first path-point P_1 (3).

$$\begin{pmatrix} x_{P,j} - x_{A/B,j} \\ y_{P,j} - y_{A/B,j} \end{pmatrix} = \begin{pmatrix} \cos(\varphi_{2,1j}) & -\sin(\varphi_{2,1j}) \\ \sin(\varphi_{2,1j}) & \cos(\varphi_{2,1j}) \end{pmatrix} \begin{pmatrix} x_{P,1} - x_{A/B,1} \\ y_{P,1} - y_{A/B,1} \end{pmatrix} \tag{3}$$

The property that the length of each link does not change can be expressed by equation (4).

$$\left(x_{A/B,j} - x_{A0/B0}\right)^2 + \left(y_{A/B,j} - y_{A0/B0}\right)^2 = \left(x_{A/B,1} - x_{A0/B0}\right)^2 + \left(y_{A/B,1} - y_{A0/B0}\right)^2 \tag{4}$$

The specification of nine path-points leads to eight equations. These eight equations can be used to calculate the eight unknown locations of the revolute joint.

In comparison to the finite positions synthesis the performance and the chance to find a suitable solution is quite bad. The advantage is that the synthesis task and the boundary conditions can be specified adjustably.

3 Hybrid Dimensional Synthesis

The aim of the hybrid dimensional synthesis is the combination of the advantages of both mentioned approaches to create an adjustable as well as performant synthesis method. Unlike the approach described in [3], the following algorithm allows the specification of joint locations. Furthermore, an analytical synthesis algorithm is shown.

3.1 Value of the synthesis task

Since this hybrid approach will also have limitations concerning the maximum number of finite positions, points of a coupler curve and joint locations, it is necessary to consider the value of the synthesis task. The approach used here is distinguished from the approach mentioned in [2] to fit the special requirements of the hybrid dimensional synthesis. The used approach will be explained by the synthesis task in Fig. 3.

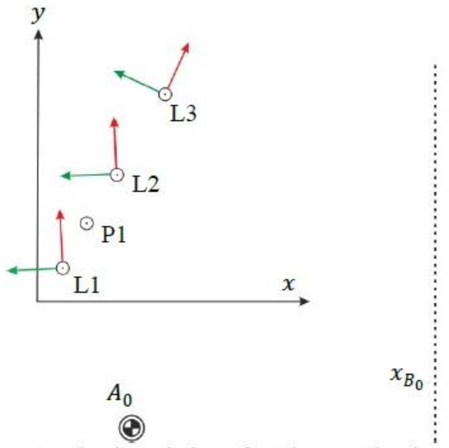

Fig. 3: Synthesis task for a four-bar mechanism

This synthesis task consists of three finite positions (L1 - L3), one point of a coupler curve (P1) and three specifications about the location of the joints (x_{Ao}, y_{Ao}, x_{Bo}). By the definition of at least one finite position or one point of the coupler curve, the remaining synthesis degree of freedom (sDOF) of a four-bar linkage is equal to the number of joints times two.

Since the finite position synthesis based on the Burmester theorem provides a synthesis method for dyads, the algorithm of identifying the sDOF has to check if the synthesis of one dyad is over-determined. The regulation for the sDOF calculation can be seen in Table 1.

A dyad based on two revolute joints (n_1 = 2) has the sDOF equals four. Each joint specification such as an x- or y-coordinate reduces the sDOF by one. The specification of the first finite position n_3 does not influence the sDOF since it defines the relative position of the coupler point. Each additional specification of a finite position n_3 reduces the sDOF by one as well.

Table 1: Calculation of the sDOF for a dyad

Dyad		*sDOF*
n_1	No. revolute joints	$+2\cdot n_1$
n_2	No. Prismatic joints	$+1\cdot n_2$
$(n_3\text{-}1)$	No. finite positions	$-1\cdot(n_3-1)$
n_4	No. joint specifications	$-1\cdot n_4$

The four-bar linkage of the synthesis task shown in Fig. 3 consists of the two dyads A_0AC and B_0BC where C represents the coupler point, that has to move through the three finite positions (L1 - L3) and point (P1). The sDOF specifications of each dyad based on Table 1 are listed in Table 2.

Table 2: sDOF specifications for A_0AC and B_0BC

A_0AC	*sDOF*
$n_1 = 2$	+4
$n_2 = 0$	0
$n_3 = 3$	-2
$n_4 = 2$	-2
Σ	**0**

B_0BC	*sDOF*
$n_1 = 2$	+4
$n_2 = 0$	0
$n_3 = 3$	-2
$n_4 = 1$	-1
Σ	**1**

It can be seen that the final sDOF of dyad A_0AC is equal to zero. There are no additional specifications possible for this dyad. The sDOF of dyad B_0BC is equal to one. The sDOF of the complete four-bar linkage is calculated by the sum of the dyads and so it is equal to one, too. Since the complete mechanism is underdetermined, there is the possibility of one additional specification concerning a desired path-point. So the overall sDOF of the synthesis task in Fig. 3 is equal to zero and can be solved exactly.

Table 3 shows the property of a synthesis task related to the sDOF. Overdetermined synthesis tasks cannot be solved exactly. They only can be approximated via an optimization. Underdetermined synthesis tasks have an infinitive number of possible solutions. They can be optimized concerning different criterions such as installation space, length of linkages, dynamical properties, transmission properties and so forth. This contribution deals with hybrid dimensional synthesis where the sDOF is equals to zero. If the synthesis task is solvable by linkages based on revolute and prismatic joints, there are always a limited number of solutions which exactly fulfill the synthesis task.

Table 3: Properties of the synthesis tasks

sDOF	*Property*
$sDOF < 0$	Overdetermined
$sDOF = 0$	Exact solution possible
$sDOF > 0$	Underdetermined

3.2 Algorithm for the synthesis

The algorithm explained within this chapter shows a new approach to synthesize these exact solutions for an sDOF equal to zero. It is necessary to distinguish between two different cases of synthesis tasks, the analytically solvable and the numerically solvable synthesis task.

3.2.1 Case 1: Analytically solvable synthesis task

The first case is the analytically solvable synthesis task. An algorithmic overview of such a task is given in Fig. 4. The specialty of this case is, that the sDOF of one dyad is equal to zero and hence can be synthesized via the finite position synthesis mentioned in chapter 2.1. The second dyad has to be underdetermined.

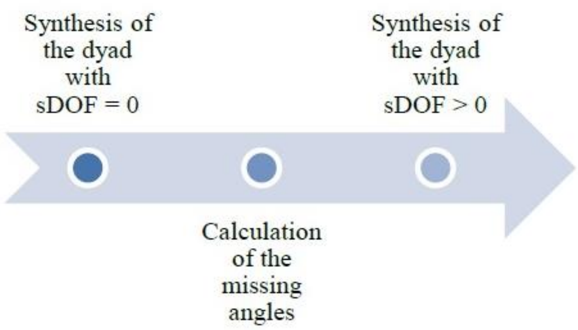

Fig. 4: Algorithm for analytically solvable synthesis tasks

To synthesize the second dyad the approach displayed in Fig. 5 based on [2] can be used. The coupler point of the synthesized dyad has to be moved in the desired path-points. Therefore it is necessary that the path-point can be reached by the dyad. Thus, the required angles of the coupler link in each point can be determined. These angles are the additional input for the synthesis of the second dyad.

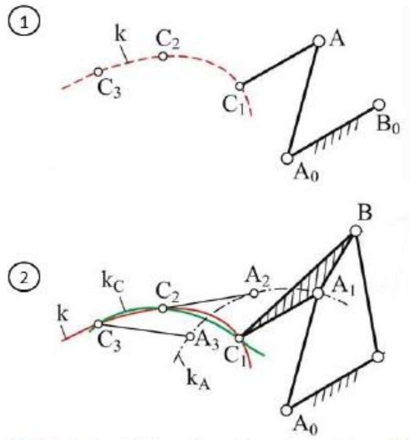

Fig. 5: Sketch of the algorithm based on [2]

To fulfill one point of the coupler curve C_i there are two possible angles γ_i of the coupler linkage (compare to Fig. 6 left).

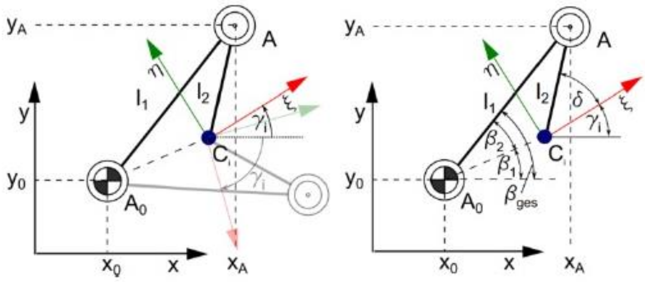

Fig. 6: Calculation of the missing angles

The angle γ_i can be calculated in subject to the already synthesized dyad A_0AC. Therefore, the angle δ can be calculated related to the position of the revolute joint A in the η,ξ-coordinate system of the coupler link:

$$\delta = \mathrm{atan2}\left(\frac{\xi_A}{\eta_A}\right) \tag{5}$$

The angles β_1 and β_2 can be calculated via equation (6) and (7) where l_C is the distance between the joint A_0 and the path-point C_i.

$$\beta_1 = \mathrm{atan2}\left(\frac{y_{Ci} - y_0}{x_{Ci} - x_0}\right) \tag{6}$$

$$\beta_2 = \mathrm{acos}\left(\frac{-l_2^2 + l_1^2 + l_C^2}{2 l_2 l_C}\right) \tag{7}$$

The angle β_{ges} now can have the two possible solutions (8) to calculate the global coordinates (9) of joint A.

$$\beta_{ges} = \beta_1 \pm \beta_2 \tag{8}$$

$$\begin{pmatrix} x_A \\ y_A \end{pmatrix} = \begin{pmatrix} l_1 \cos(\beta_{ges}) + x_0 \\ l_1 \sin(\beta_{ges}) + y_0 \end{pmatrix} \tag{9}$$

The resulting γ_i can be calculated via equation (10).

$$\gamma_i = \text{atan2}\left(\frac{y_A - y_{Ci}}{x_A - x_{Ci}}\right) - \delta \tag{10}$$

Since now every path-point of the synthesis task has at least one angle γ_i, the sDOF of the second dyad is equal to zero and can be synthesized with the finite position synthesis. This approach uses the same algorithms as the finite position synthesis and so it is as performant as the established approach. Besides, it allows a more adjustable input for the synthesis task.

3.2.2 Case 1: Example of the analytical algorithm

The synthesis task shown in Fig. 3 is an example for a analytically solvable motion task. Here the synthesis of dyad A_0AC has the sDOF equal to zero so the finite position synthesis from section 2.1 can be used. Input for this synthesis is the location of joint A_0 and the three finite positions L1 - L3. With this information the grey dyad in Fig. 7 can be synthesized.

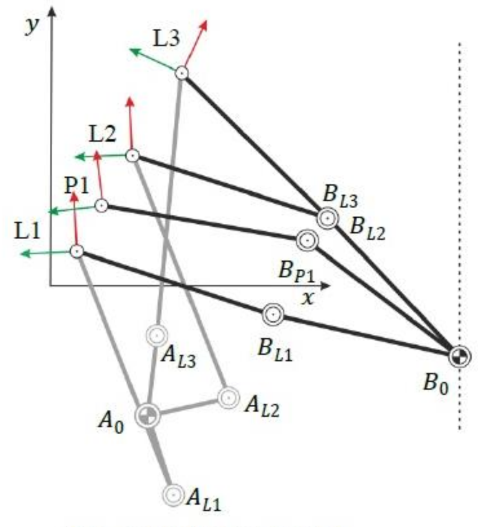

Fig. 7: Synthesis of case 1

In the next step the two possible angles γ_i of the coupler link in point P1 can be calculated via equation (5) - (10). Afterwards the next dyad can be synthesized via the input of x_{B0}, the finite positions L1 - L3 as well as the Postion P1 with the calculated angles γ_i. So this dyad is not underdetermined anymore. The black dyad in Fig. 7 shows one solution of the synthesis task for the underdetermined dyad.

Since there are two possible angles of the coupler linkage in each point, there is more than one solution for this task. Subsequently it is possible to choose from a set of solutions the best for the given synthesis task. This could be the property that all finite positions and path-points belong to the same coupler curve and thus can be reached without disassembling. This example shows the advantages of the new approach. Instead of the classical synthesis method the hybrid dimensional synthesis ensures that not only three finite positions can be specified for a given location of A_0.

3.2.3 Case 2: Numerically solvable synthesis task

If the sDOFs of both dyads are underdetermined the algorithm explained in chapter 3.2.1 is not applicable anymore. Here the recommended approach is still based on the finite position synthesis to fulfill the performance requirements. The difference is that the dyads of the linkage cannot be synthesized separately. So, the four-bar mechanism has to be considered in its entirety. For the specification of a finite position, equation (1) can be solved according to the finite position synthesis explained in section 2.1.

For the specification of a desired point of a coupler curve, the angle γ_{i+1} is not specified. Since each parameter A_j - G_j is a function of the finite positions (2) this equation system now changes if just a point of a coupler curve (x_{Cj+1}, y_{Cj+1}) is desired. The parameter A_j, D_j and E_j do not depend on the angle γ_{i+1}. All other parameters have to be itemized as shown in equation (11).

$$\begin{aligned} & A_j + D_j x_M + E_j y_M \\ & + [x_{Ci}\cos(\gamma_i) + y_{Ci}\sin(\gamma_i) - x_{Ci+1}\cos(\gamma_{i+1}) \\ & - y_{Ci+1}\sin(\gamma_{i+1})]\xi_K \\ & + [-x_{Ci}\sin(\gamma_i) + y_{Ci}\cos(\gamma_i) + x_{Ci+1}\sin(\gamma_{i+1}) \\ & - y_{Ci+1}\cos(\gamma_{i+1})]\eta_K \\ & + [-\cos(\gamma_i) + \cos(\gamma_{i+1})](x_M\xi_K + y_M\eta_K) \\ & + [-\sin(\gamma_i) + \sin(\gamma_{i+1})](y_M\xi_K - x_M\eta_K) = 0 \end{aligned} \tag{11}$$

This equation can be categorized by the terms $\cos(\gamma_{i+1})$ and $\sin(\gamma_{i+1})$ with the unknown angle γ_{i+1}. This leads to the equation (12) with the coefficients according to equation (13) - (15).

$$a_1 = a_2 \sin(\gamma_{i+1}) + a_3 \cos(\gamma_{i+1}) \tag{12}$$

$$\begin{aligned} a_1 &= -A_j - D_j x_M - E_j y_M \\ &- [x_{Ci}\cos(\gamma_i) + y_{Ci}\sin(\gamma_i)]\xi_K \\ &- [-x_{Ci}\sin(\gamma_i) + y_{Ci}\cos(\gamma_i)]\eta_K \\ &+ \cos(\gamma_i)\,(x_M\xi_K + y_M\eta_K) \\ &+ \sin(\gamma_i)\,(y_M\xi_K - x_M\eta_K) \end{aligned} \tag{13}$$

$$a_2 = -y_{Ci+1}\xi_K + x_{Ci+1}\eta_K + \xi_K y_M - \eta_K x_M \tag{14}$$

$$a_3 = -x_{Ci+1}\xi_K - y_{Ci+1}\eta_K + \xi_K x_M + \eta_K y_M \tag{15}$$

Equation (12) has to be formulated for both dyads of the four-bar mechanism and thus leads to a linear equation system (16) with two equations and the two unknowns $\cos(\gamma_{i+1})$ and $\sin(\gamma_{i+1})$. This linear equation system can be solved with Cramer's rule.

$$\begin{pmatrix} a_{2,1} & a_{3,1} \\ a_{2,2} & a_{3,2} \end{pmatrix} \begin{pmatrix} \sin(\gamma_{i+1}) \\ \cos(\gamma_{i+1}) \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ a_{1,2} \end{pmatrix} \tag{16}$$

The combination of the solution of (16) with (17) leads to the final equation (18).

$$\sin^2(\gamma_{i+1}) + \cos^2(\gamma_{i+1}) = 1 \tag{17}$$

$$\begin{aligned} &\left(a_{1,1}a_{3,2} - a_{3,1}a_{1,2}\right)^2 + \left(a_{2,1}a_{1,2} - a_{1,1}a_{2,2}\right)^2 \\ &- \left(a_{2,1}a_{3,2} - a_{3,1}a_{2,2}\right)^2 = 0 \end{aligned} \tag{18}$$

Each specification of a finite position leads to a total of two equations of type (1) (one for each dyad). Each specification of a path-point leads to one equation of type (18). Since this system of equation is no longer analytically solvable, a numerical optimization method such as a particle swarm algorithm can be used. To increase the performance at this point, the optimization algorithm will use just as many joint locations as is necessary to solve all equations of type (1). The output of this equation system is the input for the objective function (18).

3.2.4 Case 2: Example of the numerical algorithm

To show an example of the numerical algorithm the task of Fig. 8 should be synthesized.

This synthesis task consists of the definition of both joint co-ordinates of A_0, two finite positions L1 and L2 as well as four desired points of the coupler curve P1 - P4. Analogously to section 3.1 the sDOF specifications for the two dyads of this synthesis are calculated in Table 4. Each dyad has an sDOF greater than zero, so no analytical approach is possible. The sum of the sDOF of both dyads is equal to four. That enables the definition of the four desired points of the coupler curve P1 – P4.

For the two finite positions L1 and L2 one equation of type (1) per dyad can be formulated. With the information of joint A_0 the mechanism now has the sDOF of 4. Thus four parameters have to be optimized via an optimization algorithm to fulfill the four equations of type (18). A solution of the synthesis task shown in Fig. 4 can be seen in Fig. 9.

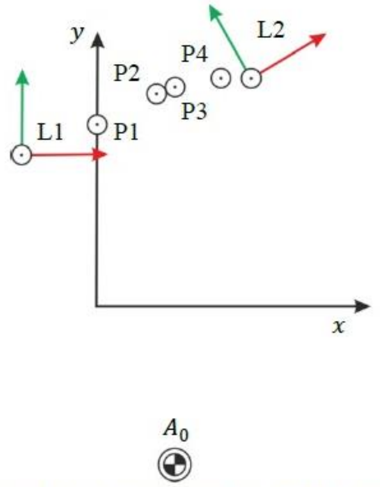

Fig. 8: Example for a numerically solvable synthesis task

Table 4: sDOF specifications for A_0AC and B_0BC

A_0AC	*sDOF*	*B_0BC*	*sDOF*
$n_1 = 2$	+4	$n_1 = 2$	+4
$n_2 = 0$	0	$n_2 = 0$	0
$n_3 = 2$	-1	$n_3 = 2$	-1
$n_4 = 2$	-2	$n_4 = 0$	0
Σ	**1**	**Σ**	**3**

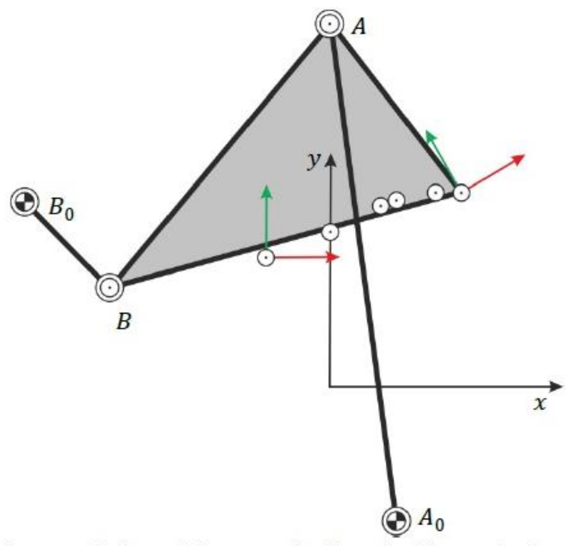

Fig. 9: Solution of the numerically solvable synthesis task

The used algorithm ensures that even if no mechanism can fulfill the synthesis task exactly, the algorithm finds an optimal solution that fulfills the finite positions exactly. The given points of the coupler curve then are just approximated as good as possible.

4 Conclusion

The shown hybrid dimensional synthesis method combines the advantages of the Burmester theorem for the synthesis of finite positions and the synthesis of path-points. This approach ensures an adjustable way of defining a synthesis task as well as a high performance of the algorithm.

By calculating the value of a synthesis task, the maximal number of finite positions, points of the coupler curve as well as joint locations can be determined. For a synthesis task that includes finite positions it is important to check the synthesis degree of freedom (sDOF) for each dyad so that no dyad is overdetermined and thus can be synthesized.

It is also shown that if one dyad of a four-bar-linkage has an sDOF equal to zero, it is possible to synthesize the mechanism analytically. Therefore the first step is the synthesis of the dyad with the sDOf equal to zero. The coupler point of this dyad now can be moved in the desired points of the coupler curve. By doing so, the possible coupler angles at these points can be calculated. They are the input for the calculation of the other dyads.

If no dyad has an sDOF equal to zero, no analytical synthesis is possible. Then the equation system based on the Burmester theorem can be modified. This modified equation system is the input for a numerical optimization algorithm. This algorithm ensures that an approximated solution can be found if no exact solution of the synthesis task is possible. The so synthesized mechanism fulfills the finite positions exactly.

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