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Particular drawing biquaternion closure equations of complex spatial mechanisms

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Abstract

This study focuses on one of the immensely important problems in the theory of mechanisms and machines - kinematic analysis of complex spatial mechanisms. The solution of this problem is related to the spatial transformations of coordinate systems. For this purpose we use quaternions, which are the most effective and versatile mathematical tools for spatial transformations. Using the principle of the transfer by Study-Kothelnikov the spatial transformation turning around point with respect to the quaternion expressions are compiled by biquaternions and these equations correspond to the transformation of coordinate system bypass circuit mechanisms. Using examined loop - closure equations for spatial 7R mechanism it were introduced the direct and inverse problem of 6R serial manipulator.

Keywords: quaternions, dual, numbers, mechanism, manipulator, transformations, analysis.

1. Introduction

The problem of analisis and sinthesis of spatial mechanisms by using quaternions algebra have been studied by several researchers. Mamedov[1] derived the formulas for relationship of quaternions with matrix mathematical apparatus for spatial transformation, and then using the principle of the transfer by Kothelnikov solved the problem of velocities and accelerations for different spatial mechanisms. F.M.Dimentberg [2] described the theory of screws, the algebra of dual numbers, performs a kinematic analysis of the spatial mechanisms on the basis of the screws algebra, describes the different groups of screws. In profound work V.N.Branets and I.P.Shmyglevskiy[3] describes in detail the shape of the quaternion algebra and their property as an operator of rotation spatial solid. The paper V.N.Branets and I.P.Shmyglevsky [3] describes in detail the shape of the algebra of quaternions and their property as an operator of the spatial turn of rigid body. It has been derived the equations of a rigid body kinematics in the quaternion presentation. The fundamental work of Kothelnikov [4] theory of screws are submitted in biquaternions presentation. It describes the essence of the important principle of mechanics - principle of "transference." The results are applied to some problems of mechanics and solid. Chevallier [5] discussed about dual quaternions in kinematics. Collins et.al. [6] studied the workspace and singular configurations of the 3- RPR parallel manipulator, where they also used quaternions. Larochelle [7] used planar quaternions to create synthesis equations for planar robots, and created a virtual reality environment that could promote the design of spherical manipulators. Martines et. al. [8] presented quaternion operators for describing the positions, angular velocity and acceletions for a spherical motion of a rigid body with respect to the reference frame. McCarthe et. al. [9] used Clifford algebra exponentials in the kinematics synthesis. Dai [10] reviewed theoretical development of rigid body displacement where he also mentions about quaternions and biquaternions. Roy et. al.[11] used quaternion interpolation in the finite element approximation of geometrically exact beam. Zupan [12] tried to implement rotational quaternions into the geometrically exact three dimensional beam theory and novel finite element formulation was proposed. Pennestri et.al. [13] used dual quaternions for the analysis of rigid body motions and tries to the kinematic modeling of the human joints. Cellodoni et.al. [14] investigated an elastic model of rod and carried out the group of rotations by using quaternions. Banavar et.al.[15] developed an analytical model of a novel spherical robot by using quaternion algebra. Liao et.al. [16] used biquternions in the inverse kinematic analysis of general 6R manipulators.

2. A brief note about quaternions.

Quaternion is a complex number made up of the real unit *1* and three imaginary units $\bar{\boldsymbol{i}}_1, \bar{\boldsymbol{i}}_2, \bar{\boldsymbol{i}}_3$ with real elements:

$$\boldsymbol{\lambda} = 1\lambda_o + \lambda_1\bar{\boldsymbol{i}}_1 + \lambda_2\bar{\boldsymbol{i}}_2 + \lambda_3\bar{\boldsymbol{i}}_3 \qquad (1)$$

Terms of multiplying the following units:

$$1\circ\bar{\boldsymbol{i}}_1 = \bar{\boldsymbol{i}}_1\circ 1 = \bar{\boldsymbol{i}}_1,\ 1\circ\bar{\boldsymbol{i}}_2 = \bar{\boldsymbol{i}}_2\circ 1 = \bar{\boldsymbol{i}}_2,\ 1\circ\bar{\boldsymbol{i}}_3 = \bar{\boldsymbol{i}}_3\circ 1 = \bar{\boldsymbol{i}}_3,$$
$$\bar{\boldsymbol{i}}_1\circ\bar{\boldsymbol{i}}_1 = -1,\ \bar{\boldsymbol{i}}_2\circ\bar{\boldsymbol{i}}_2 = -1,\ \bar{\boldsymbol{i}}_3\circ\bar{\boldsymbol{i}}_3 = -1,$$
$$\bar{\boldsymbol{i}}_1\circ\bar{\boldsymbol{i}}_2 = -\bar{\boldsymbol{i}}_2\circ\bar{\boldsymbol{i}}_1 = \bar{\boldsymbol{i}}_3,\ \bar{\boldsymbol{i}}_3\circ\bar{\boldsymbol{i}}_1 = -\bar{\boldsymbol{i}}_1\circ\bar{\boldsymbol{i}}_3 = \bar{\boldsymbol{i}}_2,$$
$$\bar{\boldsymbol{i}}_2\circ\bar{\boldsymbol{i}}_3 = -\bar{\boldsymbol{i}}_3\circ\bar{\boldsymbol{i}}_2 = \bar{\boldsymbol{i}}_1,\ 1\circ 1 = 1,$$

Rules multiplying the imaginary units stored using Fig.1 : the multiplication of two unit located on the clockwise, obtained the third unit with the sign "+", while in the reverse direction unit is obtained with the sign "-".

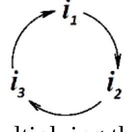

Figure 1. Rules multiplying the imaginary units

These rules indicate that the multiplication by *1* does not change the quaternion, so in the future in terms of the quaternion first term λ_o will be designated without unity.

Units $\bar{i}_1, \bar{i}_2, \bar{i}_3$ can be identified by the three-dimensional vector space and consider the coefficients of these units as a component of the vector. Accordingly, the quaternion can be represented as the sum of the scalar and vector parts:

$$\boldsymbol{\lambda} = sqal\,\boldsymbol{\lambda} + vect\,\boldsymbol{\lambda}$$

The multiplication of quaternions has associative and distributive properties with respect to addition:

$$(\boldsymbol{\lambda_1\lambda_2})\boldsymbol{\lambda_3} = \boldsymbol{\lambda_1}(\boldsymbol{\lambda_2\lambda_3}),\ \ \boldsymbol{\lambda_1}(\boldsymbol{\lambda_2} + \boldsymbol{\lambda_3}) = \boldsymbol{\lambda_1\lambda_2} + \boldsymbol{\lambda_1\lambda_3},$$

but multiplication of quaternions is not commutative. Indeed, by doing quaternion multiplication of two quaternions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ we obtain:

$$\boldsymbol{\lambda}\circ\boldsymbol{\mu} = \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \\ +\lambda_o\left(\mu_1\bar{\boldsymbol{i}}_1 + \mu_2\bar{\boldsymbol{i}}_2 + \mu_3\bar{\boldsymbol{i}}_3\right) + \\ +\mu_0\left(\lambda_1\bar{\boldsymbol{i}}_1 + \lambda_2\bar{\boldsymbol{i}}_2 + \lambda_3\bar{\boldsymbol{i}}_3\right) + \begin{vmatrix} \bar{\boldsymbol{i}}_1 & \bar{\boldsymbol{i}}_2 & \bar{\boldsymbol{i}}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \quad (2)$$

From this expression it is clear that $\boldsymbol{\lambda}\circ\boldsymbol{\mu} = \boldsymbol{\mu}\circ\boldsymbol{\lambda}$ only when disappear determinants. This is possible either when $\lambda_1 = \lambda_2 = \lambda_3 = 0$, or $\mu_1 = \mu_2 = \mu_3 = 0$, that is, when one of the factors is a scalar, or when $\boldsymbol{\lambda} = a\,\boldsymbol{\mu}$ (*a* real number). From the last expression as well we conclude that quaternion multiplication of two vectors containing the scalar and vector product of these vectors. Indeed, if in Eq.(2) to take

$\lambda_o = \mu_0 = 0$, we get:

$$\boldsymbol{\lambda}\circ\boldsymbol{\mu} = -\lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \begin{vmatrix} \bar{\boldsymbol{i}}_1 & \bar{\boldsymbol{i}}_2 & \bar{\boldsymbol{i}}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}$$

The norm of a quaternion is the product $\boldsymbol{\lambda}$ on conjugate quaternion $\tilde{\boldsymbol{\lambda}}(\lambda_o - \lambda_1\bar{\boldsymbol{i}}_1 - \lambda_2\bar{\boldsymbol{i}}_2 - \lambda_3\bar{\boldsymbol{i}}_3)$:

$$\boldsymbol{\lambda}\circ\tilde{\boldsymbol{\lambda}} = \tilde{\boldsymbol{\lambda}}\circ\boldsymbol{\lambda} = \lambda_o{}^2 + \lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2$$

This expression is obtained based on the expression (2). The norm of a quaternion is denoted by $|\boldsymbol{\lambda}|$ or λ. If $|\boldsymbol{\lambda}| = 1$, called the unity quaternion.

Any quaternion (1) may be represented by a trigonometric form:

$$\boldsymbol{\lambda} = \lambda(cos\varphi + \boldsymbol{e}\ sin\varphi)$$

where λ norm of a quaternion;

Accordingly, trigonometric unit quaternion expression will be the following:

$$\boldsymbol{\lambda} = cos\varphi + \boldsymbol{e}\ sin\varphi$$

e - is unit vector of the vector part of the quaternion $\boldsymbol{\lambda}$:

$$\boldsymbol{e} = \frac{vect\ \lambda}{\sqrt{\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2}} = \frac{\lambda_1\bar{i}_1+\lambda_2\bar{i}_2+\lambda_3\bar{i}_3}{\sqrt{\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2}}, \quad (3)$$

$$cos\varphi = \frac{\lambda_o}{\lambda_o{}^2+\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2}\ ;\ \sin\varphi = \frac{\sqrt{\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2}}{\lambda_o{}^2+\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2}$$

Turning operation.

Quaternion algebra allows us to represent a spatial transformation in a simple form. Let λ and $\boldsymbol{r}$ are non-scalar quaternions, then the value

$$r' = \lambda\circ r\circ\tilde{\lambda} \quad (4)$$

is also a quaternion scalar norm and part of which is equal to the norm and the scalar part of the quaternion ***r*** **.** Vector part *vect* $\boldsymbol{r'}$ obtained by rotating *vect* $\boldsymbol{r}$ around the cone axis by double angle 2φ **.** Operation (4) changes only the vector part of the quaternion, so that operation can be regarded as the transformation operation r of the vector into the vector $\boldsymbol{r'}$. Because the quaternion norm ***r*** does not change transformation (4), the module of the vector part ***r*** as remains unchanged. This implies that transformation (4) is orthogonal.

After completing quaternion multiplication (4) and equating the coefficients of the four units, we obtain the transformation (4) in the coordinates(for unit quaternions):

$$r_1' = (\lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2)r_1 + 2(\lambda_1\lambda_2 - \lambda_0\lambda_3)r_2 + \\ +2(\lambda_1\lambda_3 + \lambda_0\lambda_2)r_3 \\ r_2' = 2(\lambda_1\lambda_2 + \lambda_0\lambda_3)r_1 + (\lambda_0^2 + \lambda_2^2 - \lambda_1^2 - \lambda_2^2)r_2 + \\ +2(\lambda_2\lambda_3 - \lambda_0\lambda_1)r_3 \quad (5) \\ r_3' = 2(\lambda_1\lambda_3 - \lambda_0\lambda_2)r_1 + 2(\lambda_2\lambda_3 + \lambda_0\lambda_1)r_2 + \\ +(\lambda_0^2 + \lambda_3^2 - \lambda_1^2 - \lambda_2^2)r_3$$

For example, let the vector ***r*** subjected to a sequence of transformations and rotations are defined by the quaternions $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_n}$. The resulting quaternion for rotation is determined by $\boldsymbol{\lambda}$:

$$\lambda = \lambda_n\circ\lambda_{n\text{-}1}, \dots, \circ\lambda_1, \quad (6)$$

where quaternions $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_n}$ expressed in the original coordinate system. Of course, encreesing the number of successive transformations the expression (6) becomes laborious.

But if, quaternions are given as sequence of turns, using Rodrigues-Hamilton parameters, the resulting quaternion is determined by [3]:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda_1}\circ\boldsymbol{\lambda_2}\circ, \dots, \circ\boldsymbol{\lambda_n}$$

The components of the quaternion in the basis, convertible by the same quaternion, is called Rodrigues-Hamilton parameters. This quaternion components is equal in both coordinate systems because that quaternion determine the transformation from one coordinate system to another.

Dual numbers.

The dual number is as follows:

$$A = a + \delta a^0$$

where *a* - the main, a^0- moment part of the dual umber, δ - operator Clifford has property $\delta^2 = 0$. Dual numbers are denoted by basic letters. Operations on dual numbers are made according to the formulas:

$$A \pm B = (a \pm b) + \delta(a^o \pm b^o);$$

$$A\cdot B = a\cdot b + \delta(a^o b + a b^o);$$

$$\frac{A}{B} = \frac{a}{b} + \delta\frac{a^o b + ab^o}{b^2}; \quad A^n = a^n + \delta n a^o a^{n-1}; A^{\frac{1}{n}} = a^{\frac{1}{n}} + \delta\frac{a^o}{na^{\frac{n-1}{n}}}$$

The function of the dual number is as follows:

$$F(X) = f(x + \delta x^o) = f(x) + \delta x^o f'(x);$$

$$F(X, A_1, A_2, \dots, A_n) = F(x, a_1, a_2, \dots, a_n) + $$
$$+\delta\left(x^o\frac{dF}{dx} + a_1^o\frac{dF}{da_1} + a_2^o\frac{dF}{da_2} + \dots + a_n^o\frac{dF}{da_n}\right)$$

The trigonometric functions of the dual number $X = x + \delta x^o$ can be expressed as follows:

$\sin X = \sin x + \delta x^o \cos x$; $\cos X = \cos x + \delta x^o \sin x$; $tgX = tgx + \delta x^o \frac{1}{\cos^2 x}$

Dual quaternions, the transfer principle.

If in the expression (1) real numbers $\lambda_o, \lambda_1, \lambda_2, \lambda_3$ replaced by dual, we obtain an expression of the dual quaternion:

$$\Lambda = \Lambda_o + \Lambda_1\bar{i}_1 + \Lambda_2\bar{i}_2 + \Lambda_3\bar{i}_3 \quad (7)$$

where $\Lambda_к = \lambda_к + \delta\lambda_к^0$ *(k=0,1,2,3)* the components of the dual quaternion. Transform the expression (7):

$$\boldsymbol{\Lambda} = (\lambda_o + \delta\lambda_0^0) + (\lambda_1 + \delta\lambda_1^0)\bar{\boldsymbol{i}}_1 + (\lambda_2 + \delta\lambda_2^0)\bar{\boldsymbol{i}}_2 +$$
$$+(\lambda_3 + \delta\lambda_3^0)\bar{\boldsymbol{i}}_3 = \lambda_o + \lambda_1\bar{\boldsymbol{i}}_1 + \lambda_2\bar{\boldsymbol{i}}_2 + \lambda_3\bar{\boldsymbol{i}}_3 + \quad (8)$$
$$+\delta\left(\lambda_0^0 + \lambda_1^0\bar{\boldsymbol{i}}_1 + \lambda_2^0\bar{\boldsymbol{i}}_2 + \lambda_3^0\bar{\boldsymbol{i}}_3\right) = \boldsymbol{\lambda} + \delta\boldsymbol{\lambda}^0$$

Equation (8) is an expression of biquaternion. It should be noted that the expression "biquaternion" and "dual quaternion" very relative, so they are equivalent and means the same operator for most common spatial transformation. Like quaternions the biquaternion (unity) can be reduced to trigonometric form:

$$\Lambda = \cos\Phi + E\sin\Phi$$

where ***E*** **-** the unit screw of the biquaternion; *Φ* - dual argument (dual angle) biquaternion.

Like the formulas (3):

$$\boldsymbol{E} = \frac{\text{vect}\,\Lambda}{\sqrt{\Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2}} = \frac{\Lambda_1\bar{i}_1 + \Lambda_2\bar{i}_2 + \Lambda_3\bar{i}_3}{\sqrt{\Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2}}$$

$$\cos\Phi = \frac{\Lambda_o}{\Lambda_o{}^2 + \Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2};$$

$$\sin\Phi = \frac{\sqrt{\Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2}}{\Lambda_o{}^2 + \Lambda_1{}^2 + \Lambda_2{}^2 + \Lambda_3{}^2}.$$

In the fundamental paper [4] it is proved that all the formulas written for the quaternion are non-deployed biquaternion formulas. This principle is called the principle of "transference." For example, applying this principle to the rotation operation (4), we can write

$$R' = \Lambda \circ R \circ \tilde{\Lambda},$$

where screw R' is obtained by moving the screw ***R*** along the unit screw ***E*** by the double dual angle $\boldsymbol{2\Phi}$.

3. Creation of closed-loop equations of the 7R spatial mechanisms.

As is known, that the composition of close-loop equations of spatial mechanisms is a time-consuming task and the output equations of the relationship between the parameters of the mechanisms by performing multiplication in the close-loop equations is almost an impossible task for the complex spatial mechanisms with traditional spatial transformation operators, in particular using matrix form. As was shown in [1] the condition of closed form, single-loop spatial seven-bar mechanism (Figure 2) had been expressed by biquaternions product as follow:

$$\Lambda_1 \circ A_1 \circ \Lambda_2 \circ A_2 \circ \dots \circ \Lambda_7 \circ A_7 = 1 \quad (9)$$

where $\Lambda_i = \cos\Phi_k + \bar{\boldsymbol{i}}_3 \sin\Phi_k\ (k = 1,2,\dots,7)$ are biquaternions characterize movement in kinematic pairs, these biquaternions can be called as "variable", $\Phi_k = \varphi_k + \delta\varphi_k^0$ (see Figure 2);

$A_i = \cos B_k + \bar{i}_2 \sin B_k (k = 1,2,\dots,7)$ are biquaternions, characterizing link parametrers of mechanism, these biquaternions can also be called "permanent",

$B_k = \beta_k + \delta\beta_k^0$ (shown in Figure *2* for the 1st. link). In reference [1] it is shown that the equation (9) is the common for all single-loop arrangements (including the plane four-link mechanism).

Using biquaternions as operators in spatial transformation it is ability to simplify a drawing and the deployment of the loop-closure equations.

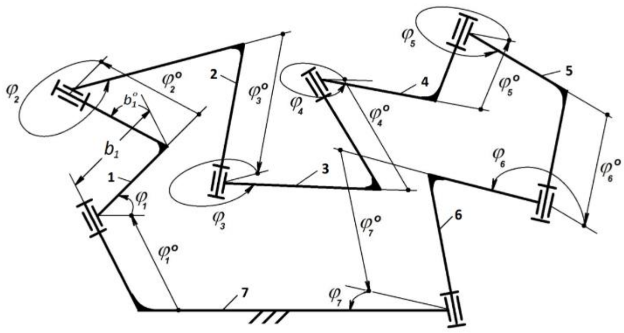

Fig. 2. The spatial 7R mechanism

Let us consider the possibility of simplifying the equations in more detail.So the equation (9) can be written as follows:

$$(\cos\Phi_1 + \bar{\boldsymbol{i}}_3 \sin\Phi_1) \circ (\cos B_1 + \bar{\boldsymbol{i}}_2 \sin B_1) \circ$$
$$(\cos\Phi_2 + \bar{\boldsymbol{i}}_3 \sin\Phi_2) \circ (\cos B_2 + \bar{\boldsymbol{i}}_2 \sin B_2) \circ \cdots$$
$$\cdots \circ (\cos\Phi_7 + \bar{\boldsymbol{i}}_3 \sin\Phi_7) \circ (\cos B_7 + \bar{\boldsymbol{i}}_2 \sin B_7) = 1$$

Biquaternions multiplication in this equation it is possible to perform a variety of options. For example, if biquaternions distributed evenly on both sides of the equation, we get:

$$\Lambda_1 \circ A_1 \circ \Lambda_2 \circ A_2 \circ \Lambda_3 \circ A_3 \circ \Lambda_4 = \tilde{A}_7 \circ \tilde{\Lambda}_7 \circ \tilde{A}_6 \circ \tilde{\Lambda}_6 \circ \tilde{A}_5 \circ \tilde{\Lambda}_5 \circ \tilde{A}_4 \quad (10)$$

where $\tilde{\Lambda}_7, \tilde{A}_7, \tilde{\Lambda}_6, \tilde{A}_6, \tilde{\Lambda}_5, \tilde{A}_5, \tilde{A}_4$ biquaternions conjugate. Disclosed, for example the left side of the biquaternion product (10), we obtain the following expression:

$$a_1(C\beta_3 C\Phi_4 + i_3 C\beta_3 S\Phi_4 + i_2 S\beta_3 C\Phi_4 + i_1 S\beta_3 S\Phi_4) +$$
$$+a_2(-C\beta_3 C\Phi_4 - i_3 C\beta_3 S\Phi_4 - i_2 S\beta_3 C\Phi_4 - i_1 S\beta_3 S\Phi_4) +$$
$$+a_3(-C\beta_3 C\Phi_4 - \boldsymbol{i_3} C\beta_3 S\Phi_4 - \boldsymbol{i_2} S\beta_3 C\Phi_4 - \boldsymbol{i_1} S\beta_3 S\Phi_4) +$$
$$+a_4(-C\beta_3 C\Phi_4 - \boldsymbol{i_3} C\beta_3 S\Phi_4 - \boldsymbol{i_2} S\beta_3 C\Phi_4 - \boldsymbol{i_1} C\beta_3 S\Phi_4) +$$
$$+a_5(-\boldsymbol{i_1} C\beta_3 C\Phi_4 + \boldsymbol{i_2} C\beta_3 S\Phi_4 + \boldsymbol{i_3} S\beta_3 C\Phi_4 - S\beta_3 S\Phi_4) +$$
$$+a_6(i_1 C\beta_3 C\Phi_4 - i_2 C\beta_3 S\Phi_4 + i_3 S\beta_3 C\Phi_4 - S\beta_3 S\Phi_4) +$$
$$+a_7(-i_1 C\beta_3 C\Phi_4 + i_2 C\beta_3 S\Phi_4 - i_3 S\beta_3 C\Phi_4 + S\beta_3 S\Phi_4) +$$
$$+a_8(-\boldsymbol{i_1} C\beta_3 C\Phi_4 + \boldsymbol{i_2} C\beta_3 S\Phi_4 - \boldsymbol{i_3} S\beta_3 C\Phi_4 + S\beta_3 S\Phi_4) +$$
$$+a_9(\boldsymbol{i_2} C\beta_3 C\phi_4 + \boldsymbol{i_1} C\beta_3 S\phi_4 - S\beta_3 C\phi_4 - \boldsymbol{i_3} S\beta_3 S\phi_4) +$$
$$+a_{10}(\boldsymbol{i_2} C\beta_3 C\Phi_4 + \boldsymbol{i_1} C\beta_3 S\Phi_4 - S\beta_3 C\Phi_4 - \boldsymbol{i_3} S\beta_3 S\Phi_4) +$$
$$+a_{11}(-\boldsymbol{i_2} C\beta_3 C\Phi_4 - \boldsymbol{i_1} C\beta_3 S\Phi_4 + S\beta_3 C\Phi_4 + \boldsymbol{i_3} S\beta_3 S\Phi_4) +$$
$$+a_{12}(-\boldsymbol{i_2} C\beta_3 C\Phi_4 + \boldsymbol{i_1} C\beta_3 S\Phi_4 - S\beta_3 C\Phi_4 + \boldsymbol{i_3} S\beta_3 S\Phi_4) +$$
$$+a_{13}(i_3 C\beta_3 C\Phi_4 - C\beta_3 S\Phi - i_1 S\beta_3 C\Phi_4 + i_2 S\beta_3 S\Phi_4) +$$
$$+a_{14}(\boldsymbol{i_3} C\beta_3 C\Phi_4 - C\beta_3 S\Phi_4 - \boldsymbol{i_1} S\beta_3 C\Phi_4 + \boldsymbol{i_2} S\beta_3 S\Phi_4) +$$
$$+a_{15}(\boldsymbol{i_3} C\beta_3 C\Phi_4 - C\beta_3 S\Phi_4 - \boldsymbol{i_1} S\beta_3 C\Phi_4 + \boldsymbol{i_2} S\beta_3 S\Phi_4) +$$
$$+a_{16}(-\boldsymbol{i_3} C\beta_3 C\Phi_4 + C\beta_3 S\Phi_4 - \boldsymbol{i_1} S\beta_3 C\Phi_4 - \boldsymbol{i_2} S\beta_3 S\Phi_4);$$

where:

$$a_1 = C\phi_1 C\beta_1 C\beta_2\, C(\Phi_2 + \Phi_3);$$
$$a_2 = C\phi_1 S\beta_1 S\beta_2\, C(\Phi_2 - \Phi_3);$$
$$a_3 = S\phi_1 C\beta_1 C\beta_2\, S(\Phi_2 + \Phi_3);$$
$$a_4 = S\phi_1 S\beta_1 S\beta_2\, S(\Phi_2 - \Phi_3);$$
$$a_5 = C\phi_1 C\beta_1 S\beta_2\, S(\Phi_2 - \Phi);$$
$$a_6 = C\phi_1 S\beta_1 C\beta_2\, S(\Phi_2 + \Phi);$$
$$a_7 = S\phi_1 C\beta_1 S\beta_2\, C(\Phi_2 - \Phi);$$
$$a_8 = S\phi_1 S\beta_1 C\beta_2\, C(\Phi_2 + \Phi_3);$$
$$a_9 = C\phi_1 C\beta_1 S\beta_2\, C(\Phi_2 - \Phi_3);$$
$$a_{10} = C\phi_1 S\beta_1 C\beta_2\, C(\Phi + \Phi_3);$$
$$a_{11} = S\phi_1 C\beta_1 S\beta_2\, S(\Phi_2 - \Phi);$$
$$a_{12} = S\phi_1 S\beta_1 C\beta_2\, S(\Phi + \Phi_3);$$
$$a_{13} = C\phi_1 C\beta_1 C\beta_2\, S(\Phi_2 + \Phi_3);$$
$$a_{14} = C\phi_1 S\beta_1 S\beta_2\, S(\Phi_2 - \Phi_3);$$
$$a_{15} = S\phi_1 C\beta_1 C\beta_2\, C(\Phi_2 + \Phi_3);$$
$$a_{16} = S\phi_1 S\beta_1 S\beta_2\, C(\Phi - \Phi_3).$$

The trigonometric expressions *sine* and *cosine* functions are represended by *S* and *C*,

After some transformations, and grouping the terms in *1*, $\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3}$, we will get:

$$C\phi_1 C\beta_1 C\beta_2 C\beta_3\, C(\Phi_2 + \Phi_3 + \Phi_4) -$$
$$-C\phi_1 C\beta_1 S\beta_2 S\beta_3\, C(\Phi_2 - \Phi_3 + \Phi_4) -$$
$$-C\phi_1 S\beta_1 S\beta_2 C\beta_3\, C(\phi_2 - \phi_3 - \phi_4) -$$
$$-C\phi_1 S\beta_1 C\beta_2 S\beta_3\, C(\phi_2 + \phi_3 - \phi_4) -$$
$$-S\phi_1 C\beta_1 C\beta_2 C\beta_3\, C(\phi_2 + \phi_3 + \phi_4) -$$
$$-S\phi_1 C\beta_1 S\beta_2 S\beta_3\, S(\phi_2 - \phi_3 + \phi_4) -$$
$$-S\phi_1 S\beta_1 S\beta_2 C\beta_3\, S(\phi_2 - \phi_3 - \phi_4) -$$
$$-S\phi_1 S\beta_1 C\beta_2 S\beta_3\, S(\phi_2 + \phi_3 - \phi_4) +$$
$$+\boldsymbol{i_1}[\, C\phi_1 C\beta_1 C\beta_2 S\beta_3\, C(\phi_2 + \phi_3 - \phi_4) -$$
$$-C\phi_1 C\beta_1 S\beta_2 C\beta_3\, S(\phi_2 - \phi_3 - \phi_4) -$$
$$-C\phi_1 S\beta_1 S\beta_2 S\beta_3\, S(\phi_2 - \phi_3 + \phi_4) -$$
$$-C\phi_1 S\beta_1 C\beta_2 C\beta_3\, S(\phi_2 + \phi_3 + \phi_4) -$$
$$-S\phi_1 C\beta_1 C\beta_2 C\beta_3\, C(\phi_2 + \phi_3 - \phi_4) -$$
$$-S\phi_1 C\beta_1 S\beta_2 C\beta_3\, C(\phi_2 - \phi_3 - \phi_4) -$$
$$-S\phi_1 S\beta_1 S\beta_2 S\beta_3\, C(\phi_2 - \phi_3 + \phi_4) -$$
$$-C\phi_1 S\beta_1 C\beta_2 C\beta_3\, C(\phi_2 + \phi_3 + \phi_4)] +$$
$$+\boldsymbol{i_2}[\, C\phi_1 C\beta_1 C\beta_2 S\beta_3\, C(\phi_2 + \phi_3 - \phi_4) - \qquad (11)$$
$$-C\phi_1 C\beta_1 S\beta_2 C\beta_3\, C(\phi_2 - \phi_3 - \phi_4) -$$
$$-C\phi_1 S\beta_1 S\beta_2 S\beta_3\, C(\phi_2 - \phi_3 + \phi_4) -$$
$$-C\phi_1 S\beta_1 C\beta_2 C\beta_3\, C(\phi_2 + \phi_3 + \phi_4) -$$
$$-S\phi_1 C\beta_1 C\beta_2 S\beta_3\, S(\phi_2 + \phi_3 - \phi_4) -$$
$$-S\phi_1 C\beta_1 S\beta_2 C\beta_3\, S(\phi_2 - \phi_3 - \phi_4) -$$
$$-S\phi_1 S\beta_1 S\beta_2 S\beta_3\, S(\phi_2 - \phi_3 + \phi_4)$$
$$-S\phi_1 S\beta_1 C\beta_2 C\beta_3\, S(\phi_2 + \phi_3 + \phi_4)] +$$
$$+\boldsymbol{i_3}[\, C\phi_1 C\beta_1 C\beta_2 C\beta_3\, S(\phi_2 + \phi_3 + \phi_4) -$$
$$-C\phi_1 C\beta_1 S\beta_2 S\beta_3\, S(\phi_2 - \phi_3 + \phi_4) +$$
$$+C\phi_1 S\beta_1 S\beta_2 C\beta_3\, S(\phi_2 - \phi_3 - \phi_4) -$$
$$-C\phi_1 S\beta_1 C\beta_2 S\beta_3\, S(\phi_2 + \phi_3 - \phi_4) -$$
$$-S\phi_1 C\beta_1 C\beta_2 C\beta_3\, C(\phi_2 + \phi_3 + \phi_4) -$$
$$-S\phi_1 C\beta_1 S\beta_2 S\beta_3\, C(\phi_2 - \phi_3 + \phi_4) -$$
$$-S\phi_1 S\beta_1 S\beta_2 C\beta_3\, C(\phi_2 - \phi_3 - \phi_4) -$$
$$-S\phi_1 S\beta_1 C\beta_2 S\beta_3\, cos(\phi_2 + \phi_3 - \phi_4)]$$

After deploying the right side of biquaternions expression (10) we obtain similar expression, which will be featured unknown angles Φ_5, Φ_6, Φ_7. Equating the terms in $\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3}, \boldsymbol{1}$ we get four dual equation. There is a dual dependence on the norm of biquaternion between these equations. Therefore, from the four dual equations only three are independent. Thus, taking any three equations of four and dividing them into the main and torque parts, we get six real equations for determining angles, $\varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7$, which are the main parts of the dual angles $\Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7$.

As it can be seen, that to make use of quaternions significantly simplify and get the ultimate expression for the loop-closure conditions of the spatial seven link mechanism, which is known as most complex single-loop mechanisms. These expressions are universal for all single-loop mechanisms.

But the most important advantage of expression (11) is that they are linear with respect to the sines and cosines angles $\Psi_1, \Psi_2, \ldots, \Psi_8$:

$$\Psi_1 = \Phi_2 + \Phi_3 + \Phi_4; \quad \Psi_5 = \Phi_5 + \Phi_6 + \Phi_7$$
$$\Psi_2 = \Phi_2 - \Phi_3 + \Phi_4; \quad \Psi_6 = \Phi_5 - \Phi_6 + \Phi_7 \qquad (12)$$
$$\Psi_3 = \Phi_2 + \Phi_3 - \Phi_4; \quad \Psi_7 = \Phi_5 + \Phi_6 - \Phi_7$$
$$\Psi_4 = \Phi_2 - \Phi_3 - \Phi_4; \quad \Psi_8 = \Phi_5 - \Phi_6 - \Phi_7$$

There are two dependencies between the unknown angles $\Psi_1, \Psi_2, \ldots, \Psi_8$:

$$\Psi_1 + \Psi_4 = \Psi_2 + \Psi_3; \quad \Psi_5 + \Psi_8 = \Psi_6 + \Psi_7 \qquad (13)$$

As a result, the mathematical model of single-loop spatial seven-bar mechanism is described by relatively simple equations, and therefore their numerical solution is not difficult.

If the terms of biquaternions leave to one side in equation (9) and perform the biquaternions multiplication, and then after equating the coefficients of the unit vectors ***1***, ***i₁***, ***i₂***, ***i₃***, we will get as unknowns the *sines* and *cosines* of the following angles:

$$\Psi_k = \Phi_2 \pm \Phi_3 \pm \Phi_4 \pm \Phi_5 \pm \Phi_6 \pm \Phi_7, \quad k = 1,2,\dots,32 \qquad (14)$$

Between angles (14) there are 26 corners dependencies such expressions (13). The resulting equations are also linear relatively unknown parameters, but in this case the number is much higher. Therefore, in the preparation of the closure equations the biquaternions advisable to distribute on both sides of the loop-closure equation.

Of course, the above very effectively and simply necessary in the preparation of loop-closure equations for platform type multi-loop mechanisms.

4. Kinematic analysis of serial 6R manipulator.

Consider the direct problem of 6R open spatial kinematic chain(Figure 3). In the direct problem it is given the set movements of the kinematic pairs. The problem is to determine the position and orientation of the gripper. Biquaternions defining the position and orientation of the rigid body is denoted by **X**:

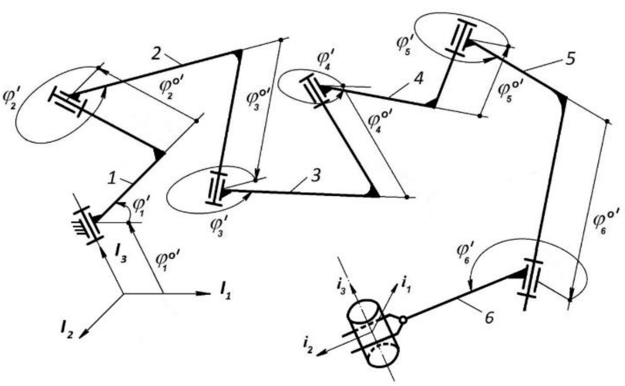

Fig. 3. Spatial 6R manipulator

$$\mathrm{X} = \mathrm{X}_o + \mathrm{X}_1\bar{\mathrm{i}}_1 + \mathrm{X}_2\bar{\mathrm{i}}_2 + \mathrm{X}_3\bar{\mathrm{i}}_3, \qquad (15)$$

Equation (15) can be expressed as the product of a quaternion:

$$\mathrm{X} = \Lambda_1 \circ \mathrm{A}_1 \circ \Lambda_2 \circ \mathrm{A}_2 \circ, \dots, \circ \Lambda_6 \circ \mathrm{A}_6 \;, \qquad (16)$$

or

$$\boldsymbol{X} = (cos\Phi_1 + \bar{\boldsymbol{i}}_3 sin\,\Phi_1) \circ (\,cosB_1 + \bar{\boldsymbol{i}}_2 \sin B_1) \circ$$
$$\circ (cos\Phi_2 + \bar{\boldsymbol{i}}_3 sin\,\Phi_2) \circ (cosB_2 + \bar{\boldsymbol{i}}_2 sin\,B_2) \circ$$
$$\circ (cos\Phi_6 + \bar{\boldsymbol{i}}_3 sin\,\Phi_6) \circ \;(cosB_6 + \bar{\boldsymbol{i}}_2 sin\,B_6)$$

where $\boldsymbol{\Phi_k}$ dual movement parameters in kinematic pairs, $\boldsymbol{B_k}$ dual number of links parameters ($k = 1,2,\dots,6$), which are discussed above. Thus, the direct problem positions of the manipulator is to implement quaternion multiplication, that can be simplified as discussed above and are described by formulas (11).

Consider the inverse problem of *6R* spatial serial manipulator. This problem can be formulated as follows. The position and orientation of the solid body(gripper) are determined by biquaternions as follow

$$\boldsymbol{X} = X_o + X_1\bar{\boldsymbol{i}}_1 + X_2\bar{\boldsymbol{i}}_2 + X_3\bar{\boldsymbol{i}}_3 =$$
$$= (x_o + \delta x_0^0) + \bar{\boldsymbol{i}}_1(x_1 + \delta x_1^0) + \bar{\boldsymbol{i}}_2(x_2 + \delta x_2^0) + \bar{\boldsymbol{i}}_3(x_3 + \delta x_3^0)$$

which determines the location of the moving coordinate system $(\bar{\boldsymbol{i}}_1, \bar{\boldsymbol{i}}_2, \bar{\boldsymbol{i}}_3)$ relative to the reference coordinate system $(\bar{\boldsymbol{I}}_1, \bar{\boldsymbol{I}}_2, \bar{\boldsymbol{I}}_3)$. Required to determine the movement in kinematic pairs, providing a predetermined position of the solid. We transform the expression (16) to the following form:

$$\boldsymbol{\Lambda_1} \circ \boldsymbol{A_1} \circ \boldsymbol{\Lambda_2} \circ \boldsymbol{A_2} \circ \boldsymbol{\Lambda_3} \circ \boldsymbol{A_3} = \boldsymbol{X} \circ \widetilde{\boldsymbol{A}}_6 \circ \widetilde{\boldsymbol{\Lambda}}_6 \circ \widetilde{\boldsymbol{A}}_5 \circ \widetilde{\boldsymbol{\Lambda}}_5 \circ \widetilde{\boldsymbol{A}}_4 \circ \widetilde{\boldsymbol{\Lambda}}_4 \qquad (17)$$

We use equation (17) in the following notation:

$$\Lambda_1 \circ \mathrm{A}_1 \circ \Lambda_2 \circ \mathrm{A}_2 \circ \Lambda_3 \circ \mathrm{A}_3 = \mathrm{M} \qquad (18)$$

$$\widetilde{\mathrm{A}}_6 \circ \widetilde{\Lambda}_6 \circ \widetilde{\mathrm{A}}_5 \circ \widetilde{\Lambda}_5 \circ \widetilde{\mathrm{A}}_4 \circ \widetilde{\Lambda}_4 = \mathrm{N} \qquad (19)$$

where

$$\mathrm{M} = \mathrm{M}_o + \mathrm{M}_1\bar{\mathrm{i}}_1 + \mathrm{M}_2\bar{\mathrm{i}}_2 + \mathrm{M}_3\bar{\mathrm{i}}_3$$
$$\mathrm{N} = \mathrm{N} + \mathrm{N}_1\bar{\mathrm{i}}_1 + \mathrm{N}_2\bar{\mathrm{i}}_2 + \mathrm{N}_3\bar{\mathrm{i}}_3$$

After completing quaternion multiplication (18), we obtain components of biquaternion M:

$$M_o = cosB_1cosB_2cosB_3\,cos(\Phi_1 + \Phi_2 + \Phi_3) -$$
$$-sinB_1sinB_2cosB_3\,cos(\Phi_1 - \Phi_2 + \Phi_3) -$$
$$-cosB_1sinB_2sinB_3\,cos(\Phi_1 + \Phi_2 - \Phi_3) -$$
$$-sinB_1cosB_2sinB_3\,cos(\Phi_1 - \Phi_2 - \Phi_3)$$

$$M_1 = -cosB_1sinB_2cosB_3\,sin(\Phi_1 + \Phi_2 - \Phi_3) -$$
$$-sinB_1cosB_2cosB_3\,sin(\Phi_1 - \Phi_2 - \Phi_3) -$$
$$-cosB_1cosB_2sinB_3\,sin(\Phi_1 + \Phi_2 + \Phi_3) +$$
$$+sinB_1sinB_2sinB_3\,sin(\Phi_1 - \Phi_2 + \Phi_3)$$

$$M_2 = cosB_1cosB_2sinB_3\,cos(\Phi_1 + \Phi_2 + \Phi_3) -$$
$$-sinB_1sinB_2sinB_3\,cos(\Phi_1 - \Phi_2 + \Phi_3) +$$
$$+cosB_1sinB_2cosB_3\,cos(\Phi_1 + \Phi_2 - \Phi_3) +$$
$$+sinB_1cosB_2cosB_3\,cos(\Phi_1 - \Phi_2 - \Phi_3)$$

$$M_3 = -cosB_1sinB_2sinB_3\,sin(\Phi_1 + \Phi_2 - \Phi_3) -$$
$$-sinB_1cosB_2sinB_3\,sin(\Phi_1 - \Phi_2 - \Phi_3) +$$
$$+cosB_1cosB_2cosB_3\,sin(\Phi_1 + \Phi_2 + \Phi_3) -$$
$$-sinB_1sinB_2cosB_3\,sin(\Phi_1 - \Phi_2 + \Phi_3).$$

After completing quaternion multiplication (19), we obtain components of biquaternion *N*:

$$N_o = \;cosB_6cosB_5cosB_4\,cos(\Phi_4 + \Phi_5 + \Phi_6) -$$
$$-sinB_1sinB_2cosB_3\,cos(\Phi_4 - \Phi_5 + \Phi_6) -$$
$$-cosB_6sinB_5sinB_4\,cos(\Phi_4 + \Phi_5 - \Phi_6) -$$
$$-sinB_6cosB_5sinB_4\,cos(\Phi_4 - \Phi_5 - \Phi_6)$$

$$N_1 = cosB_6 sinB_5 cosB_4\, sin(\Phi_4 - \Phi_5 + \Phi_6) - $$
$$-sinB_6 cosB_5 cosB_4\, sin(\Phi_4 + \Phi_5 + \Phi_6) +$$
$$+cosB_6 cosB_5 sinB_4\, sin(\Phi_4 - \Phi_5 - \Phi_6) +$$
$$+sinB_6 sinB_5 sinB_4\, sin(\Phi_4 + \Phi_5 - \Phi_6)$$
$$N_2 = cosB_6 cosB_5 sinB_4\, cos(\Phi_4 - \Phi_5 + \Phi_6) -$$
$$-sinB_6 sinB_5 sinB_4\, cos(\Phi_4 + \Phi_5 - \Phi_6) +$$
$$+cosB_6 sinB_5 cosB_4\, cos(\Phi_4 + \Phi_5 - \Phi_6) +$$
$$+sinB_6 cosB_5 cosB_4\, cos(\Phi_4 + \Phi_5 + \Phi_6)$$
$$N_3 = -cosB_6 sinB_5 sinB_4\, sin(\Phi_4 + \Phi_5 - \Phi_6) +$$
$$+sinB_6 cosB_5 sinB_4\, sin(\Phi_4 - \Phi_5 - \Phi_6) +$$
$$+cosB_6 cosB_5 cosB_4\, sin(\Phi_4 + \Phi_5 + \Phi_6) +$$
$$+sinB_6 sinB_5 cosB_4\, sin(\Phi_4 - \Phi_5 + \Phi_6)$$

We write the expression (17) with (18) and (19):

$$M = X \circ N$$

After completing quaternion multiplication and equating the terms in $\boldsymbol{1}, \boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3}$ get four dual expression:

$$M_o = X_o N_o - X_1 N_1 - X_2 N_2 - X_3 N_3$$
$$M_1 = X_o N_o - X_1 N_1 - X_2 N_2 - X_3 N_3 \qquad (20)$$
$$M_2 = X_o N_o - X_1 N_1 - X_2 N_2 - X_3 N_3$$
$$M_3 = X_o N_o - X_1 N_1 - X_2 N_2 - X_3 N_3$$

Between equations (20), there is a dual relationship to the norm biquaternion. Discarding any of them get three independent dual equations that are equivalent to six real equations. From these six equations are determined unknown corners $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6$, , that is a major part of dual angles $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6$..

Conclusions

Preparation a new method for closed-loop equations of mechanisms that particularly effective and a must in the preparation of these equations for complex multi-loop spatial mechanisms. When using offered method greatly simplified outline of the closed-loop equations of spatial mechanisms, whereby it becomes possible to express these equations in explicitly form.

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