Worksheet

(1) Describe explicitly all 2×2 row-reduced echelon matrices.

(2) Show that the system

$$\begin{aligned} x_1 - 2x_2 + x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + 7x_2 - 5x_3 - x_4 &= 3 \end{aligned}$$

has no solution.

(3) Find all solutions to the following system of equations by row-reducing the coefficient matrix.

$$\begin{aligned} x_1 + 6x_2 - 18x_3 &= 0 \\ -4x_1 + 5x_3 &= 0 \\ -3x_1 + 6x_2 - 13x_3 &= 0 \\ -7x_1 + 6x_2 - 8x_3 &= 0 \end{aligned}$$

(4) Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

For which (y_1, y_2, y_3, y_4) does the system of equations $AX = Y$ have a solution?

(5) Does the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

invertible? If A is invertible, find A^{-1}.

(6) Find the values of a, b and c for which the system

$$\begin{aligned} ax_1 + bx_2 - 3x_3 &= -3 \\ -2x_1 - bx_2 + cx_3 &= -1 \\ ax_1 + 3x_2 - cx_3 &= -3 \end{aligned}$$

has the solution $x_1 = 1,\ x_2 = -1,\ x_3 = 2$.

(7) For which values of a and b the following system has a unique solution, no solution or infinitely many solutions?

$$\begin{aligned} 2x_1 + x_2 + x_3 &= -6b \\ ax_1 + 3x_2 + 2x_3 &= 2b \\ 2x_1 + x_2 + (a+1)x_3 &= 4 \end{aligned}$$

(8) Prove that, if B is invertible then $AB^{-1} = B^{-1}A$ if and only if $AB = BA$.

(9) For an invertible matrix A, prove that $(A^T)^{-1} = (A^{-1})^T$.

(10) Let A be square matrix and $A^5 = A$. Prove that $\det(A) \in \{-1, 0, 1\}$.

(11) Find a 2×2 matrix A, such that $A^2 - 2.A - I_2 = 0$, where I_2 is the 2×2 identity matrix.

(12) Let A be an 11×11 matrix such that $A^T = -A$. Prove that $\det(A) = 0$.

(13) Let A be a square matrix and $c \neq \pm 1$ be a constant. Suppose $A^T = cA$. Prove that $\det(A) = 0$.