**MATH 265 BASIC LINEAR ALGEBRA**
**MIDTERM EXAM, 29 July 2011**
**ANSWER KEY**

| Question Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Total |
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| Question Value | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 100 |
| Student Value | | | | | | | | | | | |

1. Find the solution set of the homogeneous system

$$-x_1 + 2x_2 + x_3 - 2x_4 = 0$$
$$-x_1 - x_2 + x_3 + x_4$$
$$x_1 + x_2 + x_3 + x_4 = 0$$
$$2x_1 - x_2 - 2x_3 + x_4 = 0$$

**Answer:** The solution set of the system is:

$$S = \{ \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} \mid t \in \mathbf{R}\}$$

2. For which values of $k$ the following linear system has, a unique solution, infinitely many solution, no solution.

$$x_1 + kx_2 = 2$$
$$kx_1 + x_2 + x_3 = 1$$
$$x_1 + x_2 + x_3 = k$$

**Answer:** The system has

(a) a unique solution if $k \neq 0,\ k \neq 1$

(b) an infinitely many solution if $k = 1$

(c) no solution if $k = 0$.

3. Find the solution set of the linear system:

$$x_1 + x_2 + x_3 - x_4 = 4$$
$$x_1 - x_2 - x_3 - x_4 = 2$$
$$x_1 + x_2 - x_3 + x_4 = -2$$

**Answer:** The solution set is:

$$S = \{ \begin{bmatrix} t+3 \\ -t-2 \\ t+3 \\ t \end{bmatrix} \mid t \in \mathbf{R} \}$$

or equivalently

$$S = \{ \begin{bmatrix} t \\ 1-t \\ t \\ t-3 \end{bmatrix} \mid t \in \mathbf{R} \}$$

4. For which values of $k$ the following matrix invertible?

$$A = \begin{bmatrix} k+1 & 2 & 1 \\ 0 & 3 & k \\ 1 & 1 & 1 \end{bmatrix}$$

Find $A^{-1}$ when $k = 1$.

**Answer:** The matrix is invertible for all $k \neq 0, 4$.

For $k = 1$, $A^{-1} = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ 1/3 & 1/3 & -2/3 \\ -1 & 0 & 2 \end{bmatrix}$

5. Suppose that $A$ and $B$ are $4 \times 4$ invertible matrices. If $\det(A) = -2$ and $\det(B) = 3$ compute the following determinants.

(a) $\det(AB) = -6$

(b) $\det(A^{-1}B^2) = \frac{-9}{2}$

(c) $\det(2A^T) = -32$

(d) $\det(A^3.B^{-3}) = \frac{-8}{27}$

(e) $\det(\text{adj}(A)A^{-1}) = 4$

6. For an $n \times n$ matrix $A$ such that $\det(A) \neq 0$, prove that

(a) $[\text{adj}(A)]^{-1} = \frac{1}{\det(A)} A$.

(b) $\det(\text{adj}(A)) = \det(A)^{n-1}$.

**Answer:**

(a) It is straightforward to check that $\text{adj}(A).\frac{1}{\det(A)} A = \frac{1}{\det(A)} A.\, \text{adj}(A) = I_n$

(b) We know that $\text{adj}(A) = \det(A)A^{-1}$. Evaluate the determinant of both sides to get the required equality.

7. Consider the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

(a) Find $\det(A)$.

(b) Find $\text{adj}(A)$.

(c) Find $\det(\text{adj}(A))$.

(d) Find $A^{-1}$, if it exists.

**Answer:**

(a) $\det(A) = -4$

(b) Find $\text{adj}(A) = \begin{bmatrix} 0 & 2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}.$

(c) $\det(\text{adj}(A)) = 16$

(d) $A^{-1} = \begin{bmatrix} 0 & -1/2 & 1/2 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$

8. Use Cramer's rule to solve the system

$$2x - y + z = 3$$

$$x - 2y + 4z = 1$$

$$x - y - z = 5$$

**Answer:** $x = \frac{3}{4}$, $y = \frac{-23}{8}$, $z = \frac{-11}{8}$

9. Compute adj$(A)$ if $A$ is an invertible matrix with $A^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

   **Answer:** adj$(A) = \det(A).A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

10. For the matrix $A = \begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}$, find an upper triangular $2 \times 2$ matrix $U$ and a lower triangular $2 \times 2$ matrix $L$ such that $A = UL$.

    **Answer:** $U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 2 & 0 \\ -2 & -1 \end{bmatrix}$

    These $U$ and $L$ are not unique!!