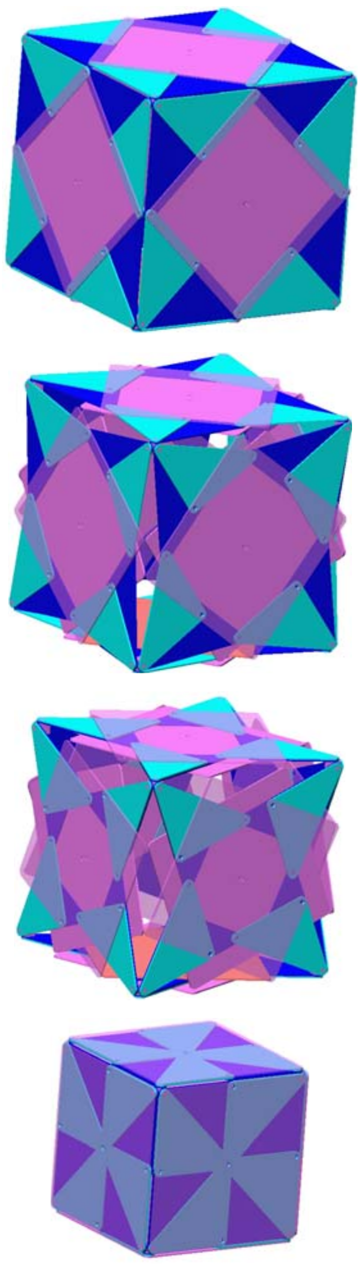
POLYHEDRA

A Historical Review



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Ankara – February 2007

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INTRODUCTION

Fire represents the tetrahedron,

air consists of octahedra,

water of icosahedra,

earth of cubes and,

while a fifth arrangement is possible,

God has used the dodecahedron,

to serve as a contour of the universe.

by Timaeus of Lokri and cited in Plato's (427-348/347) "Timaeus" [1]

Polyhedra have been focus of many people since ancient times. The subject has a special place among the subjects created by human beings by abstraction and idealization: polyhedra are harmonious and mysterious... In this study, the question of how polyhedra were treated throughout the history by different civilizations and some individuals is addressed.

The first sections are organized for different civilizations, since developments performed are strongly related to the cultural development of nations. Later on as the world becomes smaller, the studies become international, so individuals determine the main title of the sections. The latter sections include recent studies about polyhedra.

NEOLITHIC AGES

Fig 1 Platonic solids from carved stones – 2000 BC [2]

Hundreds of carved stone spheres, roughly three inches in diameter, believed to date to around 2000 BC, have been found in Scotland. Some are carved with lines corresponding to the edges of regular polyhedra. Roughly half have 6 knobs---like the one at right above---but the others range from 3 to 160 knobs. The more mathematically regular ones do not appear to have had a special importance. For example, in addition to the 12-knob dodecahedral form shown in the center and just to its right above, there are also ones with 14 knobs, corresponding to a form with two opposite hexagons, each surrounded by six pentagons. Nonetheless, the dodecahedron appears here long before the Greeks wrote of it. The function of these stones is unknown. The material varies from easily carved sandstone and serpentine to difficult, hard granite and quartzite [2]. However, notice that the third and fourth stones both represent the dodecahedron and the icosahedral appearance of the fourth stone is a trick. Also the second stone does not represent the tetrahedron but a compound polyhedron: the tetrahedron together with its dual (actually its dual is itself – see section "Greeks" for information about duals).

EGYPTIANS (2650 – 500 BC)

Fig 2 The Great Pyramid – Cheops [3]

Egyptians were surely much interested in polyhedra. Many mysteries about the pyramids they have built remain unexplained still in our time.

Probably, the most striking information about Egyptians' approach to mathematics problems is that they have used lots of examples instead of general formulae. For instance, a papyrus, which is now exhibited in Courtesy of the State Pushkin Museum of Fine Arts, reads [4]

> Method of calculating a truncated pyramid.
> If you are told: a truncated pyramid of 6 cubits in height,
> Of 4 cubits on the base, by 2 on top
> You are to square this 4: result 16
> You are to double this 4: result 8
> You are to square 2: result 4
> Add together this 16, the 8 and the 4: result 28
> Take 1/3 of 6: result 2
> Take 28 twice: result 56
> See, it is 56 – you have found it right.

Letting h, a, b represent the height, base and top measurements for a truncated square pyramid, the above description yields the volume formula

$$\frac{1}{3} h\left(a^2 + ab + b^2\right)$$

There is not much information about Egyptian's works today, but from the papyri in hand it is known that they were very experienced in calculating volumes, areas and slopes of two dimensional Figs and pyramids.

BABYLONIANS (2000 – 500 BC)

Babylonians used similar descriptions as Egyptians for calculation of volumes of solid objects. They multiplied base area with the height for prisms. However, the formulae they used for pyramids and truncated pyramids were not all true. For example, they would multiply the height of a truncated pyramid by the average of the base and top areas to find the volume. As Egyptians, Babylonians too used examples instead of symbolized formulae [5].

CHINESE

The interest of Chinese was also volumes of polyhedra, specifically prisms, pyramids and truncated pyramids. But they used general formulae as oppose to Egyptians and Babylonians. Although the resources of information about ancient Chinese's work on polyhedra are not elder than 2000 years, it is supposed that the origins of these works lie in much more ancient times.

Just as Egyptians and Babylonians, Chinese would not necessarily prove the correctness of their methodology, up to third century AD. One can see the first systematical attempts to prove the arguments among the Chinese mathematicians in **Liu Hui**'s *Commentary on the Nine Chapters* (263 AD). Liu Hui explains in his work that the *Nine Chapters* (written before third century BC), is an old text containing 246 mathematics problems. Liu Hui's contribution to this text is mainly the proofs [5].

Liu Hui assumes the volume of a rectangular parallelepiped as the product of its three dimensions and finds the other polyhedra's volumes using his four blocks: cube, qiandu, yangma, bienao (Fig 3). First he finds the other three blocks' volumes using the cube, and then he uses the four blocks to find the volume of the polyhedra of interest [6].

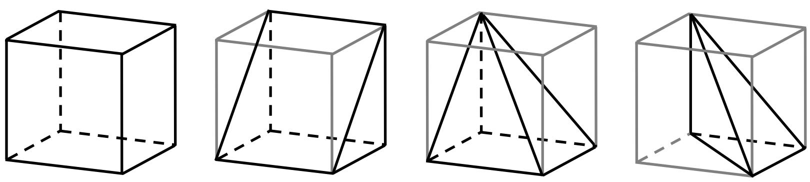

Fig 3 cube, qiandu, yangma and bienao [5]

It is easy to show that a cube can be cut into two qiandus. Also a cube dissects into three yangmas or six bienaos (a bieano has half the volume of a yangma) (Fig 4). Fig 5 is an example of his illustrations.

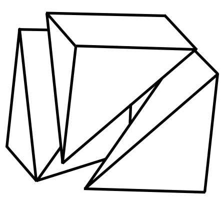

Fig 4 The cube dissected into three yangmas [5]

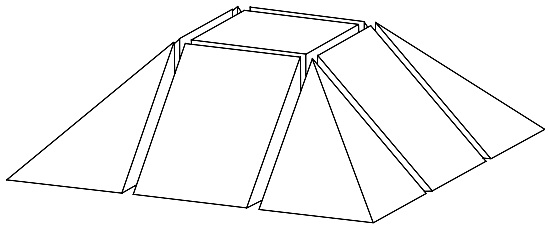

Fig 5 A truncated square pyramid dissected into a cube, four qiandus and four yangmas [5]

Fig 5 illustrates a truncated square pyramid with base side length $a = 3$, top side length $b = 1$ and height $h = 1$. Liu Hui shows that $abh + b^2h + h = 3$ (cube + 4 qiandus + 4 yangmas). Hence, the formula $\frac{1}{3} h\left(a^2 + ab + b^2\right)$ holds for the volume of the frustum.

Liu Hui's dissection method may also be used to show that the formula holds for any truncated square prism, but he seems not to be interested in the general case [5].

Among his other proofs, the most striking one is the derivation of the volume of a pyramid. He makes use of repeated dissections [6]. As a consequence of a theorem proved by Max Dehn in 1900 that any proof of the volume of a pyramid must use infinitesimal considerations in one form or another, Liu Hui does in fact use a limit process [6]! It is amazing that he considered these methods by himself at his time.

GREEKS

There are many thinkers that worked on polyhedra among the ancient Greeks. However, focus on Plato's work will be kept in this document.

Early civilizations worked out mathematics as problems and their solutions. According to B. L. van der Waerden there are so many similarities between the studies of Egyptians, Babylonians, Chinese and also Indians that he believes that these different civilizations' work originate from a common source. He proposes that all these cultures are affected by studies carried on in the Neolithic Age, say between 3000 and 2500 BC [7]. However, ancient Greeks' approach is totally different: proofs are indispensable parts of analyses.

Cromwell, in his *Polyhedra*, mentions the probability that Greek mathematicians, who liked traveling a lot, needed proofs to decide whether Babylonians' methods or Egyptian's methods are the true ones [5].

Fig 6 Plato (427- 347 BC) [8]

Plato, when the concern is polyhedra, is most well known as the comments on the five regular polyhedra, which are named after him. However, Plato was not the first to recognize them. Pythagoreans already knew three of them for their regularity: the cube, the tetrahedron (they would call it a pyramid) and the dodecahedron. **Theaetetus**, a friend of Plato, is known to first discover the regularity the icosahedron and the octahedron. It must be emphasized that these solids were already known to people, but Pythagoreans and Theaetetus were the ones discovering their regularity. Plato's contribution to the subject was not discovering the regular polyhedra, but associating them to the elements constructing the world [5] (Fig 7):

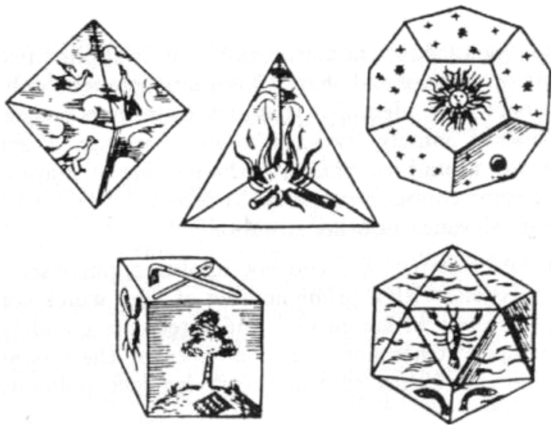

Fig 7 Johannes Kepler's interpretation of the Platonic solids [5]

> *To earth, then, let us assign the cubic form, for earth is the most immovable of the four and the most plastic of all bodies, and that which has the most stable bases must of necessity be of such a nature. Now, of the triangles which we assumed at first, that which has two equal sides is by nature more firmly based than that which has unequal sides, and of the compound Figs which are formed out of either, the plane equilateral quadrangle has necessarily a more stable basis than the equilateral triangle, both in the whole and in the parts. Wherefore, in assigning this Fig to earth, we adhere to probability, and to water we assign that one of the remaining forms which is the least movable, and the most movable of them to fire, and to air that which is intermediate. Also we assign the smallest body to fire, and the greatest to water, and the intermediate in size to air, and, again, the acutest body to fire, and the next in acuteness to air, and the third to water. Of all these elements, that which has the fewest bases must necessarily be the most movable, for it must be the acutest and most penetrating in every way, and also the lightest as being composed of the smallest number of similar particles, and the second body has similar properties in a second degree, and the third body, in the third degree. Let it be agreed, then, both according to strict reason and according to probability, that the pyramid is the solid which is the original element and seed of fire, and let us assign the element which was next in the order of generation to air, and the third to water. We must imagine all these to be so small that no single particle of any of the four kinds is seen by us on account of their smallness, but when many of them are collected together, their aggregates are seen. And the ratios of their numbers, motions, and other properties, everywhere God, as far as necessity allowed or gave consent, has exactly perfected and harmonized in due proportion.*
>
> by Plato in *Timaeus*, p1181 [9]

Ancient Greeks believed that the physical world was made up of four basic elements and their combinations: fire, air, water and earth. Fascinated by the various beautiful aspects of the regular polyhedra, Plato imagined a world consisting of them. With his own reasoning he assigned each of the regular polyhedra to a basic element. But there still remains one polyhedron out when four of them are assigned to the four basic elements. Plato associated the remaining polyhedron, the dodecahedron, to the universe, and named a fifth element: ether [9].

This association plays great role in Plato's *Timaeus*, the article he has written for his friend Theaetetus, who died after having serious injuries in a battle. His theses were so harmonic and smooth that this vision of the cosmos affected many philosophers, mathematicians and artists (Fig 8). The Lord's perfect world had to be constructed with perfect geometrical shapes:

> *As God brought into being the celestial virtue, the fifth essence, and through it created the four solids . . . earth, air, water, and fire ... so our sacred proportion gave shape to heaven itself, in assigning to it the dodecahedron . . . the solid of twelve pentagons, which cannot be constructed without our sacred proportion. As the aged Plato described in his Timaeus.*

By Pacioli, L. in *De Divina Proportione*, 1509 [9]

The «sacred proportion» Pacioli refers to is the golden ratio (The edges of a dodecahedron can be obtained by placing three mutually orthogonal rectangles having golden ratio as the ratio of the side lengths in a symmetric manner).

Fig 8 The Sacrament of the Last Supper by **Salvador Dali** [10]

Because of his work about the five regular polyhedra, Plato is known as an early scientist proposing an atomic model for the matter [5].

At this point, a conjugation property of polyhedra shall be mentioned: duality. Most basically, the *dual* of a polyhedron is obtained by replacing the faces of a polyhedron with vertices and vice versa. Theoretically all the polyhedra have duals, but not all are finite polyhedra. The duals can be obtained by connecting geometric centers of the faces, resulting a new polyhedron inside. This operation is illustrated below for the tetrahedron, the cube and the icosahedron:

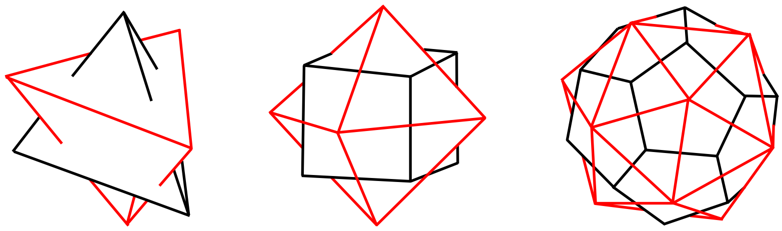

Fig 9 Duals of Platonic solids – tetrahedron-tetrahedron pair, cube- octahedron pair, icosahedron-dodecahedron pair

The dual of the dual of a polyhedron is itself. So duality is a conjugation. From Fig 9, one can see that duals of Platonic solids are again Platonic solids. Tetrahedron is self-dual, while cube-octahedron and icosahedron-dodecahedron pairs are duals of each other. A polyhedron combined with its dual constitutes a compound polyhedron. The compounds for the Platonic solids are given in Fig 10.

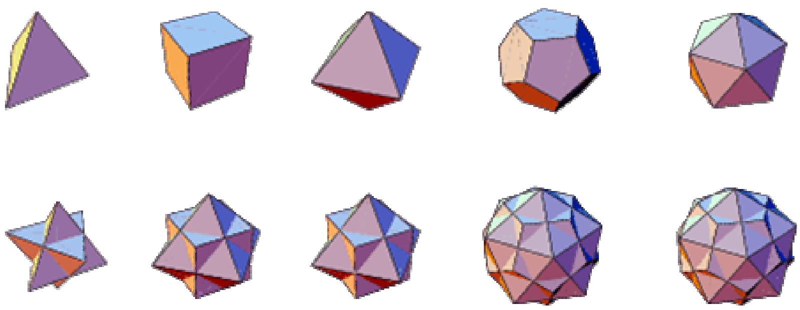

Fig 10 The compounds for the Platonic solids [11]

Two duals also can be obtained from each other by means of truncations and expansions (snubbing/extension/augmentation). Proper truncations (cuttings of pyramids on each vertex) or expansions (assembling pyramids on each face) of dual polyhedra give the same polyhedron [12] (Fig 11). For the dual polyhedra P and P', this fact is shown in Fig 12 (P1-

P1’, P2-P2’ and tP-eP are duals of each other; P1-P1’, P2-P2’ are forms in between P, P’ and tP, eP). The truncation/expansion series of Platonic solids are illustrated in Fig 13-15.

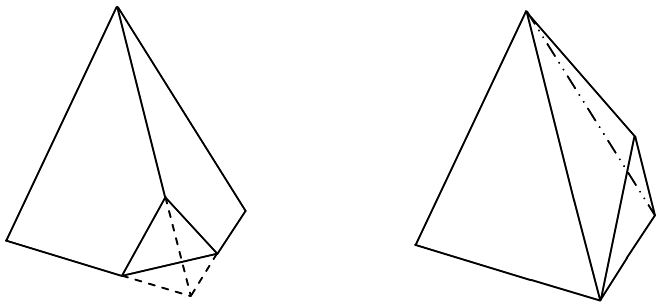

Fig 11 Truncation and expansion

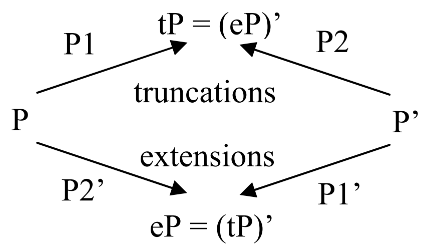

Fig 12 Truncation/expansion sequence diagram [12]

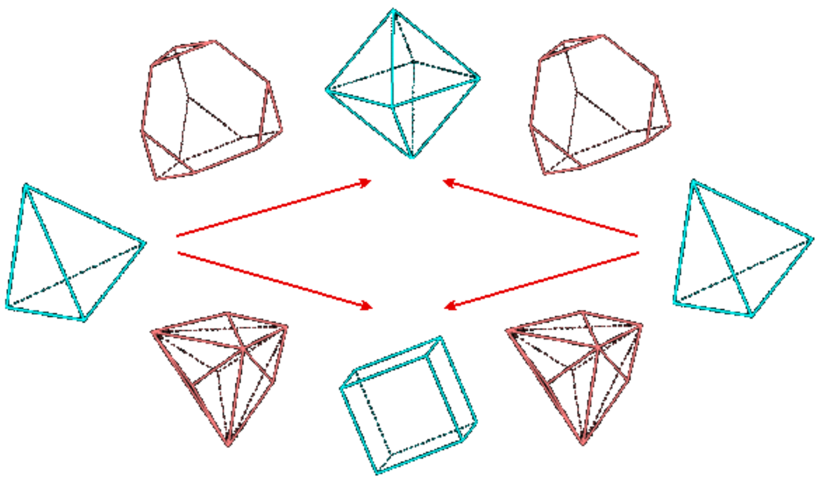

Fig 13 Tetrahedron-to-tetrahedron (tP = octahedron, eP = cube) [12]

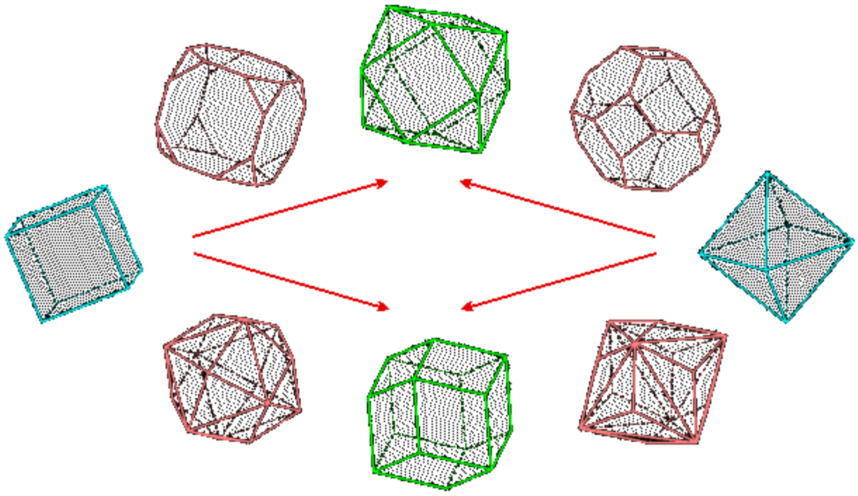

Fig 14 Cube-to-octahedron (tP = cuboctahedron, eP = rhombic dodecahedron) [12]

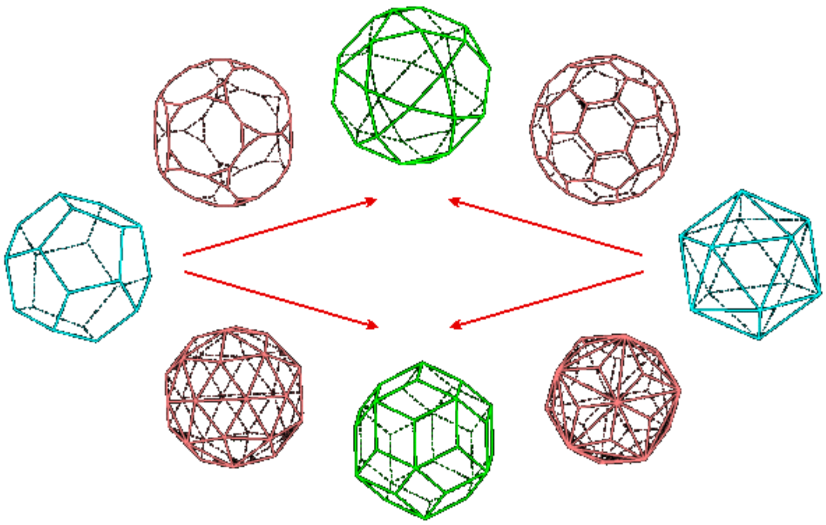

Fig 15 Dodecahedron to icosahedron (tP = icosidodecahedron, eP = rhombictriacontahedron) [12]

The seven non-Platonic truncated Figs shown in Figs 13-15 are Archimedean solids and the corresponding duals are the Catalan solids (See Section "Alexandrians" for detailed information).

Plato did not give a proof that there are only five regular polyhedra. Actually, he did not even formally specify the properties a regular polyhedron must satisfy. The first of the proofs was given by an Alexandrian mathematician: Euclid.

ALEXANDRIANS

Fig 16 Euclid (325 – 265 BC) [13]

Alexandria was the center of scientific investigations of its time; and the most popular researchers of Alexandria are Euclid and Archimedes. **Euclid** proves that there are no regular polyhedra other than the five Platonic solids as a remark at the end of 18^{th} proposition of 13^{th} book of his *Elements* [14]. His claim also defines what a regular solid is: *no other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to one another*. Euclid's definition of regular polyhedra is, however, incomplete. It would be complete if he also included the condition that each vertex should join equal number of faces. There exist five more polyhedra satisfying Euclid's original definition: five of the deltahedra. Deltahedra are equilateral triangle faced convex polyhedra. There are totally eight convex deltahedra; three of which are regular polyhedra (tetrahedron, octahedron, icosahedron, triangular dipyramid, pentagonal dipyramid, tri-augmented triangular prism, gyro-elongated square dipyramid, Siamese dodecahedron).

Euclid's proof is very straightforward, simple and short: he just analyses the possible number of possible regular polygons that can meet at a vertex and comes up with the only five possibilities. In addition to this proof, Euler has more than twenty propositions relating polyhedra. He, like Liu Hui, sometimes uses dissections (Fig 17).

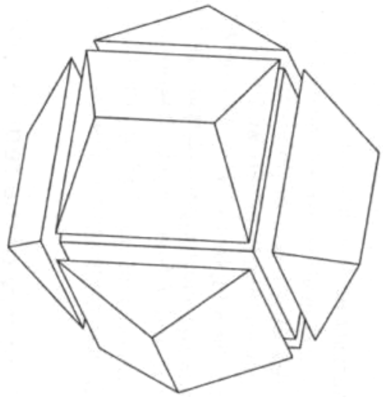

Fig 17 Euler's construction of the dodecahedron by placing roofs on faces of a cube [5]

Fig 18 Archimedes (287 – 212 BC) [15]

Archimedes is a Greek mathematician and engineer born and died in Sicily, but he has probably studied in Alexandria for a long period. He is, to many mathematicians, one of the three greatest mathematicians of all time, Isaac Newton and Carl Friedrich Gauss being the other two [16]. The thirteen semi-regular polyhedra are named after him. A key characteristic of the Archimedean solids is that each face is a regular polygon, and around every vertex, the same polygons appear in the same sequence (Fig 19).

The Archimedean solids, somewhat, can be derived using the Platonic solids. Nine of them can be obtained by truncation of a Platonic solid (Fig 20), and two further can be obtained by a second truncation. The remaining two solids, the snub cube and snub dodecahedron, are obtained by moving the faces of a cube and dodecahedron outward while giving each face a twist [17]. The duals of the Archimedean solids are called the Catalan solids (named after the Belgian mathematician **Eugéne Catalan** - 1865) (Table 1).

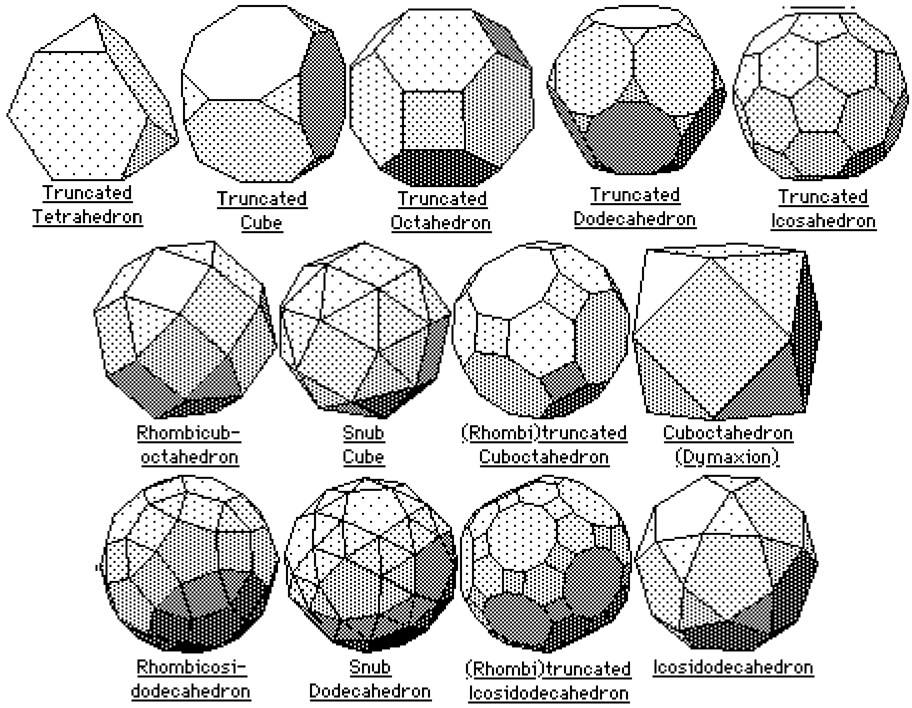

Fig 19 The thirteen Archimedean solids [18]

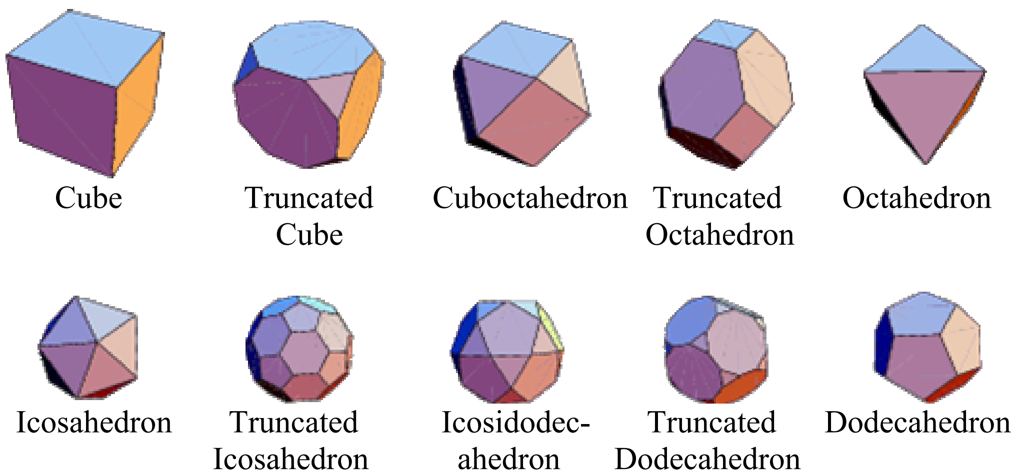

Fig 20 Cube→octahedron and icosahedron→dodecahedron series [19]

cuboctahedron	rhombic dodecahedron
great rhombicosidodecahedron	disdyakis triacontahedron
great rhombicuboctahedron	disdyakis dodecahedron
icosidodecahedron	rhombic triacontahedron
small rhombicosidodecahedron	deltoidal hexecontahedron
small rhombicuboctahedron	deltoidal icositetrahedron
snub cube	pentagonal icositetrahedron
snub dodecahedron	pentagonal hexecontahedron
truncated cube	Small triakis octahedron
truncated dodecahedron	triakis icosahedron
truncated icosahedron	pentakis dodecahedron
truncated octahedron	tetrakis hexahedron
truncated tetrahedron	triakis tetrahedron

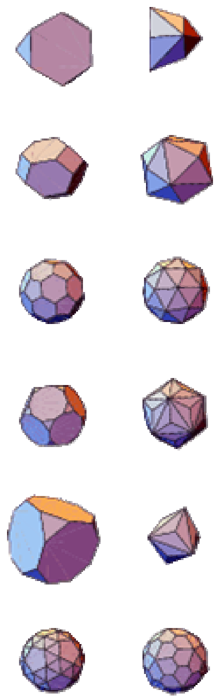

Table 1 Arcimedean solids and the corresponding Catalan solids [20]

Some mathematicians had argued that there is one more semi-regular polyhedron: the elongated square gyrobicupola (Fig 21). But, today it is known that this solid does not belong to the set that the thirteen Archimedean solids constitute because of lacking the symmetry level the other solids have.

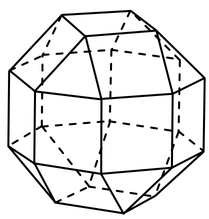

Fig 21 The elongated square gyrobicupola [5]

ARABIANS

After the rise of Islam, the center of science and knowledge moved to Baghdad. Many Arabian mathematicians worked on polyhedral geometry, however, the development is not very noticeable. Thabit ibn Qurra (836 – 901) and Abu'l-Wafa (940 - 998) are two of the mathematicians worked on polyhedra [5].

BACK TO THE WEST

European people rediscovered the foundations of scientific knowledge through the crusades. 11th and 12th centuries were times of translations and new ideas and publications started to arise in the 13th century [5]. However, no great progresses can be noted until the 16th century, the century in which perspectives started to be popular. Polyhedra were now, the frequently used tool of art. Some of the plates are given in Fig 22-25 (See [1] for some details).

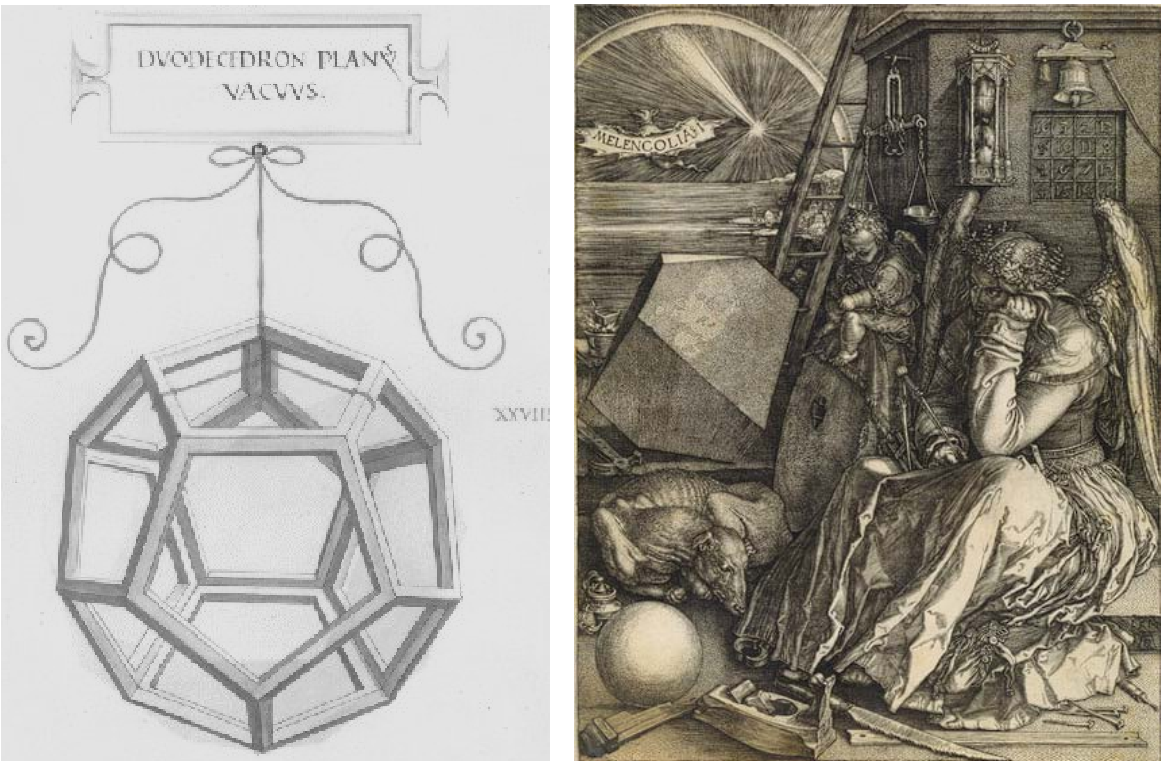

Fig 22 From *Divina Proportione* of **Luca Pacioli** by **Leonardo da Vinci** [21] and the famous engraving *Melancholia* by **Albrecht Dürer** [22]

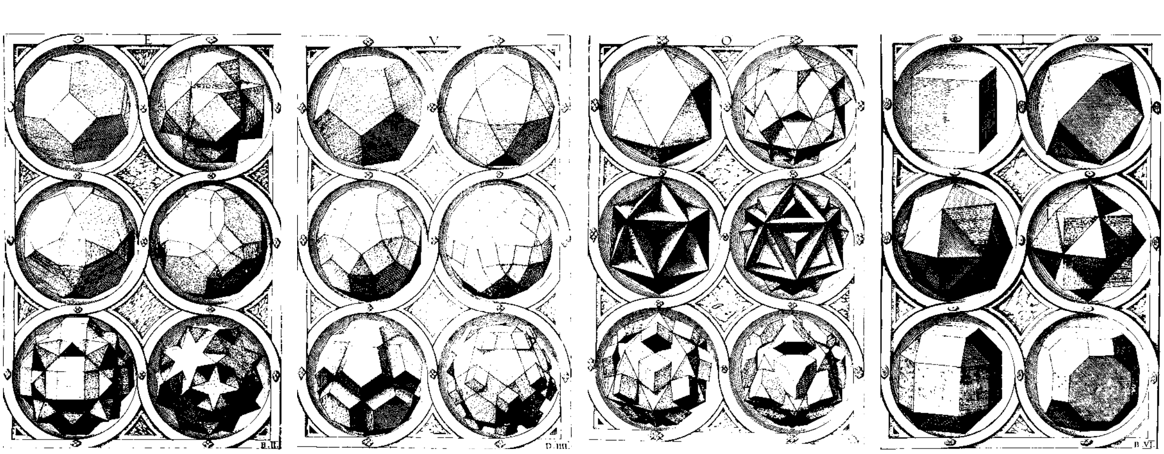

Fig 23 From *Perspectiva Corporum Regularium* by **Wenzel Jamnitzer** [23]

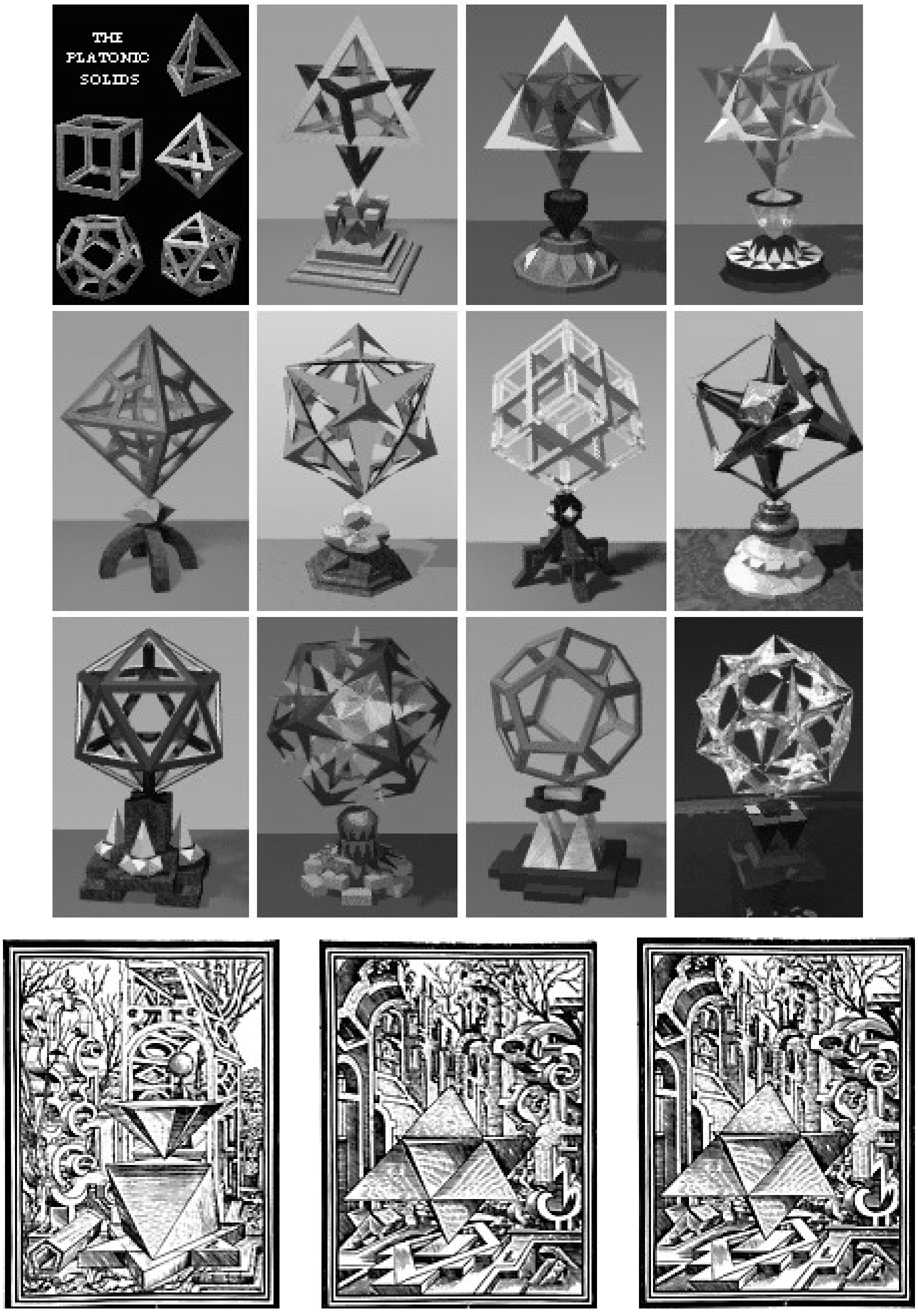

Fig 25 From *Perspectiva Corporal Regularium* by **Wenzel Jamnitzer** [23]

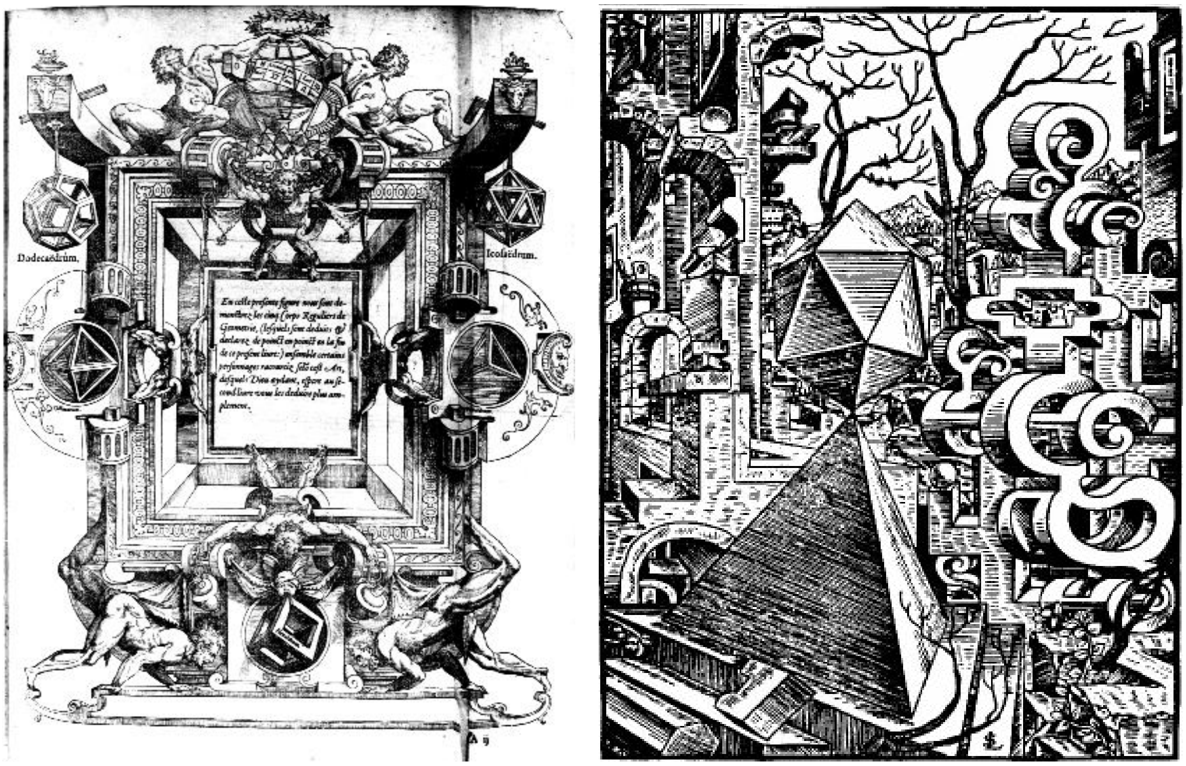

Fig 23 From *Livre de Perspective* by **Jean Cousin** [24] and from *Geometria et Perspectiva* by **Lorenz Stoer** [25]

JOHANNES KEPLER

Fig 26 Johannes Kepler (1571 - 1630) [26]

Johannes Kepler is an astronomer besides being a mathematician. He is best known by his studies relating the orbits of the planets of the solar system. In his book *Mysterium Cosmographicum*, Kepler tries to explain the order in the universe by use of observations and mathematics [5, 27]. At the time only six of the planets in the solar system were known. Kepler relates these six planets to the five regular solids as follows [28]:

> *We must first eliminate the irregular solids because we are only concerned with orderly creation. There remain six bodies, the sphere and the five regular polyhedra.*

To the sphere corresponds the outer heaven. On the other hand, the dynamic world is represented by the flat-faced solids. Of these there are five. When viewed as boundaries, however, these five boundaries determine six distinct things – hence the six planets that revolve about the sun.

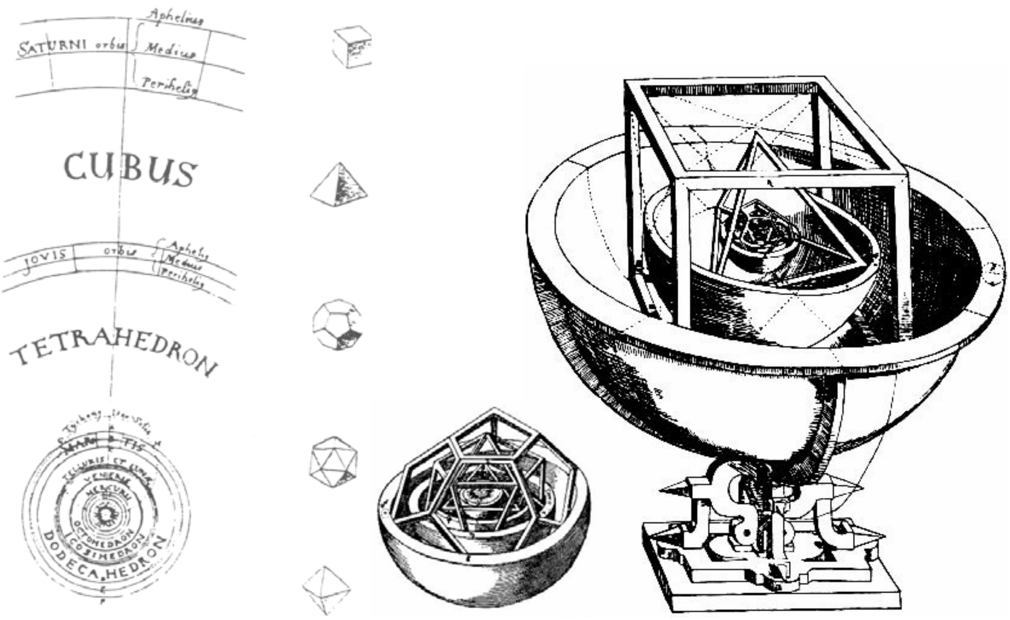

Fig 27 From *Mysterium Cosmographicum* by Johannes Kepler [26]

Kepler's approach to the structure of the universe is similar to Ancient Greeks' in that his starting point is that the structure is based on perfect geometric figures and proper ratios. With this belief he works on perfect figures and reveals many fascinating properties of polyhedra. He starts with a classification of polyhedra (Fig 28). Rhombic polyhedra are the ones to be mentioned for the first time by Kepler. Kepler describes two of such figures: the rhombic dodecahedron and the rhombic triacontahedron (these two polyhedra will show themselves as duals of two Archimedean solids: the cuboctahedron and the icosidodecahedron) (Fig 29).

$$\overbrace{\overbrace{\begin{matrix}\text{Regular}\\ \text{(Platonic)}\end{matrix}\quad\begin{matrix}\text{Half-Regular}\\ \text{(Rhombic)}\end{matrix}}^{\begin{matrix}\text{Most Perfect}\\ \text{(congruent faces)}\end{matrix}}\quad\overbrace{\text{Archimedean}\quad\begin{matrix}\text{Imperfect}\\ \text{(Prismatic)}\end{matrix}}^{\begin{matrix}\text{Perfect to a Lower Degree}\\ \text{(regular faces of several kinds)}\end{matrix}}}^{\text{Perfect (Similar vertices)}}$$

Fig 28 Kepler's Classification of Polyhedra [5]

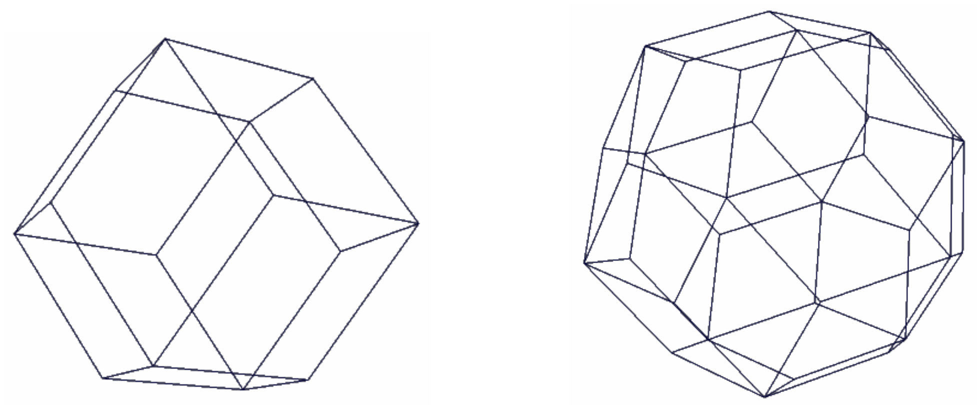

Fig 29 Kepler’s rhombic polyhedra: rhombic dodecahedron and rhombic triacontahedron [5]

Kepler is also known as the first mathematician who discovered the Archimedean solids after Archimedes. He constructs the thirteen solids in a vertex-based systematic method.

A final note about Kepler’s work on polyhedra can be the non-convex star polyhedra. These polyhedra have many relationships with convex polyhedra and are known as Kepler – Poinset solids.

RENÉ DESCARTES

Fig 30 René Descartes (1596-1650) [29]

Just like the other revolutions Descartes made in many scientific areas, the new approach he imposed in polyhedral geometry is a radical attempt. Descartes is the first scientist who explored polyhedra in general and deduced the properties of special polyhedra as special cases of the general results he obtained. His work on polyhedra gave birth or influenced many branches of mathematics.

Especially one theorem of Descartes about polyhedra (presented in his *Progymnasmata de Solidorum Elementis* [30]) is charming: the sum of deficiencies of the solid angles in a polyhedron is eight right angles (The proof follows immediately from the *Spherical Excess* formula). This theorem is "A very beautiful and general theorem which ought to be placed at the head of the theory of polyhedra" according to E. Prouhet [5].

For polyhedra, the *solid angle* is a quantity assigned to a vertex (Fig 31). It is the area of the unit sphere portion corresponding to the vertex. Its unit is *steradians*. The angle by which the sum of the plane angles around a solid angle is less than 2π is called its *deficiency* [5].

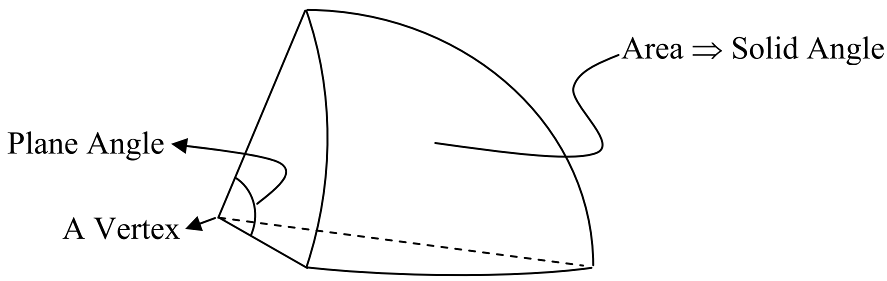

Fig 31 Vertex, plane angle and solid angle

One of the corollaries of Descartes' theorem is that there can be only five regular polyhedra. Consider a polyhedron with *V* vertices, each surrounded by *S* faces of each having *n* equal length sides. Then the sum of interior angles of a face is $(n - 2)\pi$ and so, every plane angle measures $(n - 2)\pi/n$. *S* plane angles meet at *V* vertices, so the sum of the plane angles is $SV(n - 2)\pi/n$. By the theorem, $\left(2\pi - \frac{(n-2)\pi}{n}S\right)V = 8\frac{\pi}{2} \Rightarrow V = \frac{4n}{2(n+S)-nS}$. The denominator can be factorized as $4 - (n - 2)(S - 2)$, which implies $(n - 2)(S - 2) < 4$. Then, possible integer pairs for (n, S) are then (3, 3), (3, 4), (3, 5), (4, 3) and (5, 3), which describes the tetrahedron, the octahedron, the icosahedron, the cube and the dodecahedron, respectively and uniquely. As opposed to Euclid's proof, Descartes' proof is algebraic in nature.

LEONARD EULER and HIS FORMULA

Fig 32 Leonard Euler (1707 - 1783) [31]

Polyhedra seem to be forgotten for over a century after Descartes. Euler was the mathematician drawing attention on the polyhedra back. In a letter to Goldbach in 1750, Euler writes [32]

> *Recently it occurred to me to determine the general properties of solids bounded by plane faces, because there is no doubt that general theorems should be found for them, just as for plane rectilinear figures, whose properties are:*
>
> *(1) that in every plane figure the number of sides is equal to the number of angles, and*
>
> *(2) that the sum of all the angles is equal to twice as many right angles as there are sides, less four.*
>
> *Whereas for plane figures only sides and angles need to be considered, for the case of solids more parts must be taken into account.*

When he started studying, he probably found the terminology deficient, so he created his own terminology. In his first paper on polyhedra, he defines the characteristics of the geometry as follows [33]:

> *Three kinds of bounds are to be considered in any solid body; namely points, lines and surfaces, or with the names specifically used for this purpose: solid angles, edges and faces. These three kinds of bounds completely determine the solid.*

Euler is the first mathematician to consider the edges of a polyhedron. Today, we use two of the terms: edges and faces, but instead of the solid angle, we use vertex: a term due to **Arthur Cayley** [5].

Among many relations Euler derived for polyhedra, the most important one is

$$V + F = E + 2$$

which is known as the *Euler's Formula*. V, F and E refer to the numbers of vertices, faces and edges, respectively. This formula is valid for most of the polyhedra, but not all. The formula finds application in many fields. First of all, it is an indispensable, frequently used formula in graph theory. Also mechanical engineers use the formula for planar mechanisms.

One of many consequences of the formula is the proof that there exist only five regular polyhedra. Suppose, a regular polyhedron has V vertices, E edges and F faces with each having n sides. Also let S faces meet at each vertex. Then, nS sides come together to construct the polyhedron. Two sides construct an edge when faces are assembled, so $nS = 2E$. Also each edge has two ends, resulting $SV = 2E$. Substituting $F = 2E/n$ and $V = 2E/S$ in the formula one has $E = \frac{2nS}{2(s+S)-nS}$. Note that substituting $V = 2E/S$ in this equation gives the result Descartes derived. By the same discussion Descartes has, the only possible integer pairs for (n, S) are (3, 3), (3, 4), (3, 5), (4, 3) and (5, 3).

Also one can express the number of faces and vertices in terms of n and S: $F = \frac{4S}{2(n+S)-nS}$ and $V = \frac{4n}{2(n+S)-nS}$ (Descartes' deduction). By substituting the possible (n, S) values into these formulae one obtains the corresponding face and vertex number for regular polyhedra. With this terminology, the shortest list of conditions for being regular is obtained: faces should have equal number of sides and same number of faces must meet at each vertex. Notice that no requirements for equality of angles exist in the list.

Another consequence of Euler's formula, together with two other inequalities, is the complete set of possible (V, F) pairs for polyhedra. The inequalities are as follows: every face has at least three sides, so $2E \geq 3F$ and at least three faces meet at a vertex, so $2E \geq 3V$. The formula with these two inequalities yields two bounds for (V, F) pairs: $V \geq F/2 + 2$ and $2F - 4 \geq V$.

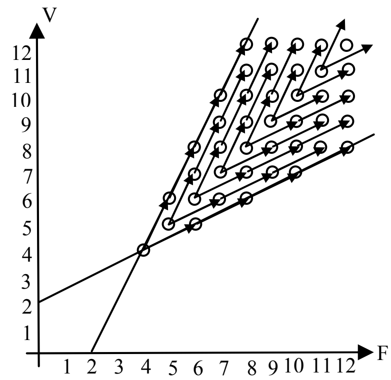

Fig 33 Possible (V, F) pairs for polyhedra [5]

Drawing the two line bounds one can obtain the possible (V, F) pairs as in Fig 33. It's possible to show that at least one polyhedron corresponds to each circle. First of all, every n-sided polygon based pyramid has $n + 1$ faces and $n + 1$ vertices. So the pyramids are examples for the circles on the diagonal. Examples for the other circles can be found by either truncation or expansion. But, expansion should be performed carefully if the resulting polyhedron is to remain convex.

Truncating a pyramid at a vertex, where three faces meet, adds two to the number of vertices and one to the number of faces. Also, this truncation results in at least one vertex meeting three faces, so repeated truncations may be applied infinitely. Expansion on one of the triangular faces of a pyramid increases the number of faces by two and number of vertices by one. Also, this process can be applied infinitely many times, provided that the polyhedron remains convex. The polyhedron needs to remain convex for that some nonconvex polyhedra do not satisfy Euler's formula.

The validity of Euler's proof was based on repeated truncations, however, repeated truncations cannot be applied to all polyhedra.

The next proof was given by **Adrien Marie Legendre** in 1794. Legendre made use of radial projections of polyhedra on spheres. He performed the proof only for convex polyhedra. Later, in 1810, **Louis Poinsot** showed that Legendre's proof also applies to nonconvex polyhedra that can be radially projected onto a sphere. The interesting point in Legendre's proof is that he uses metric properties of a sphere to prove the invariancy of quantities relating a polyhedron [5].

Next mathematician providing a proof for Euler's formula is **Augustin Louis Cauchy**. His proof, published in 1813, does not rely on metric properties. A new notion, *deformability*, introduced by Cauchy is used in this proof. He starts by choosing a face on the polyhedron and shrinks the other faces' vertices on the plane properly. Then he shows that $V + F = E + 1$ for this planar network. Again, there are polyhedra for which this method fails to verify Euler's formula [5].

In the first half of the 19th century, some exceptions to Euler's formula were noted by scientists. Notification of these exceptions resulted in alternative formulae valid for larger set of polyhedra. This search brought a necessity for a definition of a polyhedron that can be certified by everyone. Making this definition required some serious effort [34]:

> *What makes the theory of polyhedra very difficult is that it requires an essentially new science, which may be called 'geometry of position' because its principal concern is not the size or proportion of figures, but the order and (relative) position of the elements composing them.*

Indeed a new area, now called *topology*, developed in the second half of the 19th century. Cromwell describes this reform as follows [5]:

> *And indeed a new discipline was born out of the struggle to find the foundations on which the formula rested – a discipline related to geometry as algebra is related to arithmetic. It concentrates on the relationships and connections between the various constituent elements; specific details such as size, area, angles, and in fact all metric properties are ignored, just as algebraic equations express general relationships between numbers but do not deal with particular cases.*

People started to construct the terminology of this new science, and some objects were redescribed by this new terminology. Tunnels through solids were analyzed by means of non-separating curves (a closed curve in a surface such that the surface remains in one piece) and cavities in the solids were thought to be resulting disconnected surfaces. And new definitions for polyhedra arose: **August Ferdinand Möbius** defined a polyhedron as a system of polygons arranged in such a way that the sides of exactly two polygons meet at every edge and it is possible to travel from the interior of one polygon to the interior of any other without passing through a vertex [5]. This new vision of polyhedra was well accepted, but the definition still needed to be worked out, because it could not avoid some singular cases. The definition used today is a slightly modified version of Mobius' definition [5]:

A *polyhedron* is the union of a finite set of polygons such that

- Any pair of polygons meet only at their sides or corners.
- Each side of each polygon meets exactly one polygon along an edge.
- It is possible to travel from the interior of any polygon to the interior of any other.
- Let V be any vertex and let F_1, F_2, …, F_n be the n polygons meeting at V. Then it is possible to travel over the polygons F_i from one to any other without passing through V.

First condition excludes star polyhedra of the kind described by Poinsot and other self intersecting polyhedra. Second and fourth conditions exclude singular edges and vertices, and third condition ensures that the polyhedron is connected. So, among many counterexamples to Euler's formula, only one class remains: the polyhedra having tunnels. [5]

With this definition, Euler's formula can be modified as $V - E + F = 2 - 2g$, where g is, roughly speaking, the number of tunnels through the polyhedron (See [5] for the details). Formally, g is called the *genus* of the polyhedral surface and is a topologically invariant property of a surface defined as the largest number of nonintersecting simple closed curves that can be drawn on the surface without separating it. $V - E + F$ is also given a special name: *Euler characteristic* of the surface.

A complete proof for the Euler's formula was given by **Karl Georg Christian von Staudt** in his *Geometrie der Lage* (1847) [35]. His related theorem is as follows:

Let P be a polyhedron (as defined above) such that

- any two vertices are connected by a path of edges, and
- any closed curve on the surface separates P into two pieces.

Then P satisfies Euler's Formula: $V + F = E + 2$.

A corollary of this theorem is that if a polyhedron satisfies Euler's formula, it can be deformed into a sphere, and vice versa.

Another great use of Euler characteristic is noted in the **Gauss-Bonnet** theorem: Suppose that a connected network on a smooth surface *S* has *V* vertices, *E* edges and *F* faces and Gaussian curvature at a point on the surface is *k*. Then

$$\int_S k\ dA = 2\pi\left(V - E + F\right)$$

The striking point of the theorem is that it relates metric quantities to topological properties. This theorem shows that the total curvature is independent of the geometry. On the other hand, keeping the surface fixed and altering the network keeps the Euler characteristic.

SYMMETRY GROUPS

Symmetries of polyhedra have been explored mainly by physicists, chemists and mathematicians. A symmetrical polyhedron is characterized by the fact that it looks the same from different viewpoints. To describe the different kinds of symmetry, it is helpful to investigate the operations which carry a polyhedron into its indistinguishable positions. Such an operation is called a symmetry of the polyhedron [5].

There are two types of symmetries that can be considered in three dimensions: *rotation* and *reflection* symmetries (and combinations of these two). Rotation symmetry can be considered as a direct symmetry, i.e. repositioning the object determines the symmetry operation. For reflection symmetry, one needs the use of mirrors. For a rotation symmetry, an axis determines the set of points that remain fixed, whereas the fixed points lie on a plane for the reflection symmetry (See the appendix for the symmetry groups of polyhedra).

Mathematicians developed group theory while investigating the possible symmetries of polyhedra. Searching for the symmetries, people started to express the symmetries and combinations of these symmetries for a polyhedron by tables. Later they noticed some rules about these tables, and these rules gave birth to the abstract object, group.

Arthur Cayley first noticed the properties that symmetry structures satisfied: closedness, inverses, identity element and associativity (1854). **William Rowan Hamilton**, in 1856, gave a method for describing groups without writing out the complete group table. **Camille Jordan** was the first mathematician to use the term group for symmetry structures (1869). To Jordan, groups were sets closed under some operation. The four modern axioms of a group were first published in 1882 independently by **Walter von Dyck** and **Heinrich Weber**. Group theory is widely studied and applied today in many branches of science, such as particle physics, molecular bonding schemes in chemistry, classification of patterns and ornamental designs, the description of different kinds of geometry and crystallography [5].

CRYSTALLOGRAPHY

The interest to symmetry groups mostly originated from the researches about the structure of crystals. First, the crystals were thought to have spherical building blocks (**Robert Hooke** – 1665). **Christian Hauygens** was the one that suggested lattice structures for crystals (17^{th} century). The building blocks with flat faces were first proposed by **Domenico Guglielmini** (late 1600s). **René Just Haüy** (1743 - 1822) developed the flat faced building blocks idea to an extend that he is now known as the father of crystallography [5].

The lattice structure of a crystal restricts the kinds of rotational symmetry that can appear to 2-fold, 3-fold, 4-fold and 6-fold (a rotation of $(360/n)^o$ is called an n-fold rotation). Hence, there are only finitely many possibilities for the symmetry types of external forms of crystals: these 32 symmetry types are called the *crystal classes* [5].

H. S. M. COXETER - THREE DIMENSIONS DO NOT SUFFICE

As topology further developed and abstract algebra improved, three dimensional objects did not satisfy geometers and they started defining fictitious objects in higher dimensions. After polygons of two dimensions and polyhedra of three dimensions, a general term for all dimensions was defined: *polytopes*.

Harold Scott Macdonald Coxeter (1907 - 2003) is the dominant mathematician working on polyhedra in 20^{th} century. His foundations in geometry are still being studied widely. He has lots of contributions to the new branch of geometry: polytopes (Fig 34). He is also known for his work on non-Eulidean geometries.

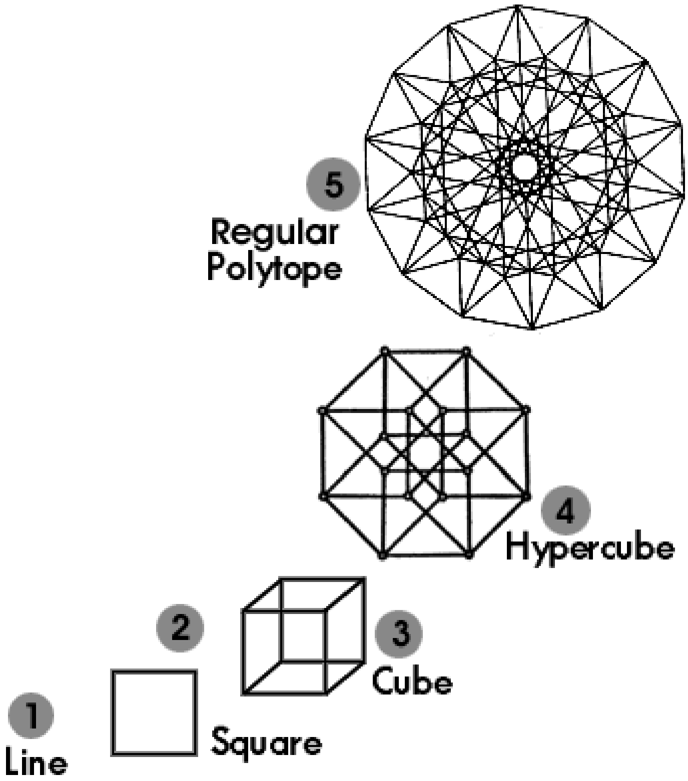

Fig 34 Projections of some polytopes to plane [36]

TESSELLATIONS

In 1900, **David Hilbert** (1862 - 1943) proposed a total of 23 problems about different areas of mathematics. 18th of these problems was about building spaces with congruent polyhedra. Later, this topic has been widely studied and is still being studied.

Regular tiling of polygons or polyhedra (or polytopes in general) is called a *tessellation*. Tessellations in two dimensions are abundantly studied, but tessellations in higher dimensions still need to be extensively explored. A recent study about space-filling polyhedra belongs to Kara Joy Duckett [16]. Some spatial tessellation examples of Ducket are given in Fig 35.

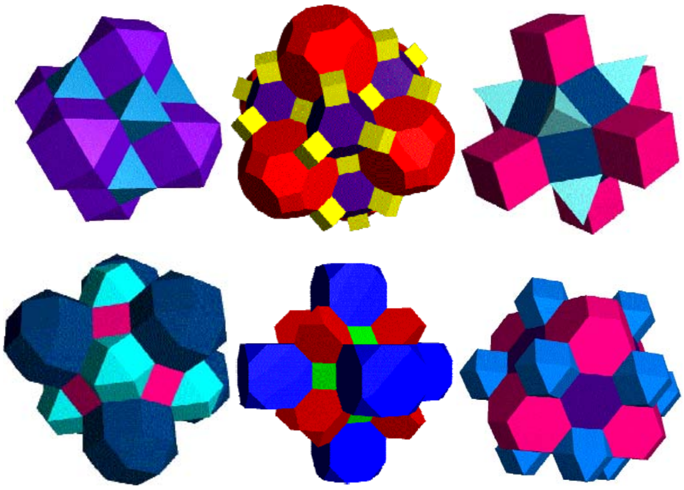

Fig 35 Examples of Ducket's tessellation units [16]

Another mathematician working on space-filling polyhedra is Guy Inchbald. Three of his original space-filling polyhedra (the bisymmetric hendecahedra, the sphenoid hendecahedra and the rhombic dodecahemioctahedron) are given in Fig 36.

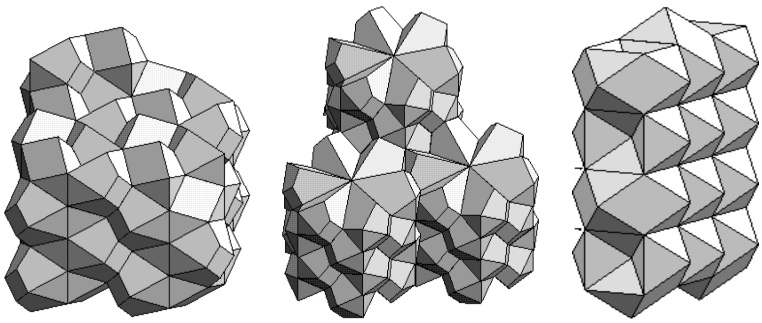

Fig 36 Three of Inchbald's tessellations [37]

CARTOGRAPHY

Polyhedra are frequently used in modeling the map of earth. The best model created up to now is **Buckminster Fuller**'s Dymaxion Map (1956) [38]. The map is an unfolded icosahedron (Fig 37).

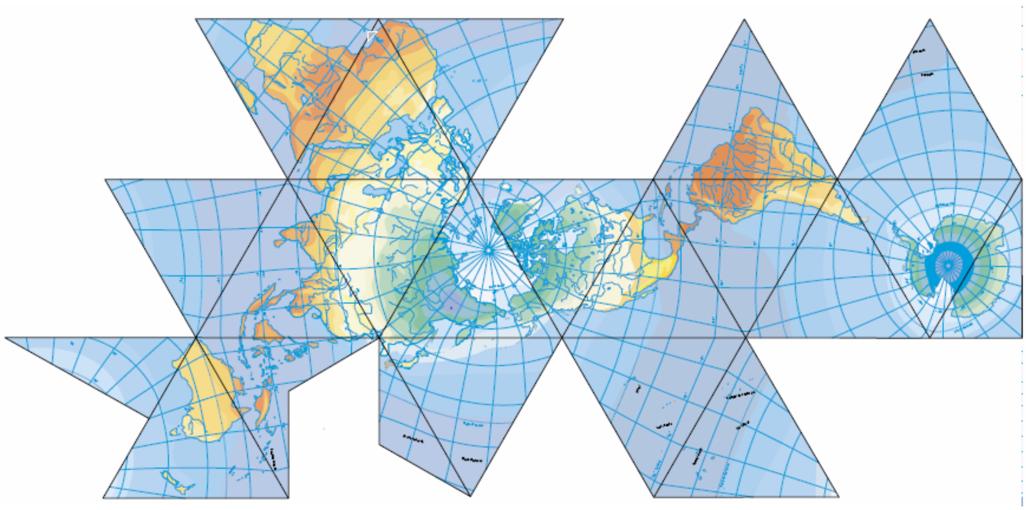

Fig 37 Fuller's Dymaxion Map [38]

FOLDING and UNFOLDING

Folding and unfolding of polyhedra is an old research area, however, systematical work about the subject is recent. The main problems that are being handled are which polyhedra can be unfolded to construct polygons and how; which polygons can be folded to construct polyhedra and how; which polyhedra can be unfolded and refolded to form other polyhedra and how. The problems are also being extended to higher dimensions. One may find a lot of useful information and links to other sites in **Eric D. Demaine**'s folding and unfolding page (Fig 38) [39].

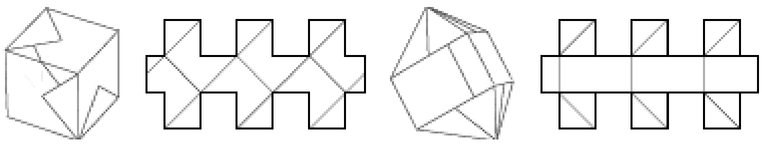

Fig 38 Unfolding a cube to fold an elongated triangular diprism [39]

VIRUSES and the EXPANDOHEDRA

Recently, microbiologists asked for aid from mathematicians and mechanical engineers about motion mechanisms of viruses. It was already known that some viruses had polyhedral outer structures (these viruses are named as *polyhedral viruses*). Now scientists explore their motion by means of solid models. A Hungarian group of scientists' examples of expandable polyhedra (expandohedra, as they call) are illustrated in Fig 39-40.

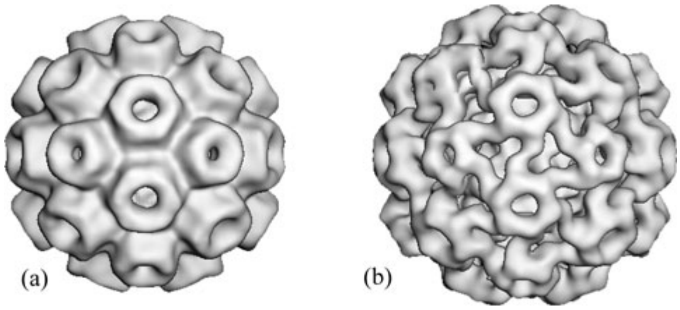

Fig 39 A realistic model for the motion of cowpea chlorotic mottle virus [40]

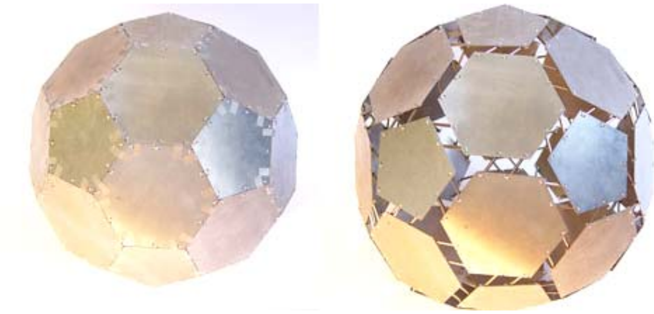

Fig 40 A mechanical model for the motion of cowpea chlorotic mottle virus [41]

POLYHEDRAL LINKAGES

In the need for combining the fascinating geometry of polyhedra and motion, inventors and engineers found a new area of study for themselves: polyhedral linkages. Not much has been done about the subject yet, however, it seems that the subject will be of concern for many years.

Engineers had a lot of beautiful examples before they started working on the subject: An American investigator's, Chuck Hoberman's, amazing toys were ready to work on (Fig 41-42) [42].

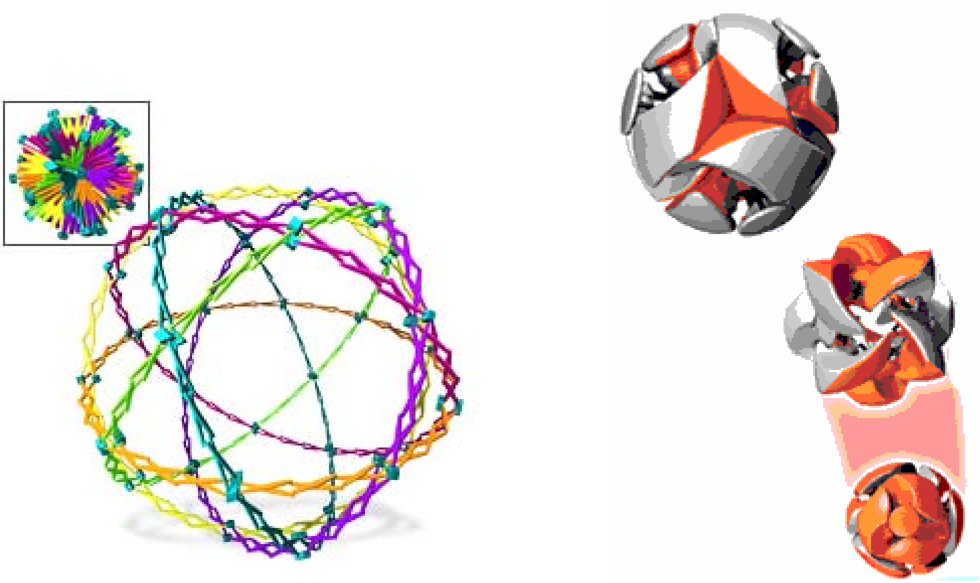

Fig 41 Two of the Hoberman toys [42]

Hoberman also have architectural products. The huge domes he designed are well accepted all around the world (See [42] for more examples and animations).

Fig 42 The iris dome and the fabric dome of Hoberman Associates [42]

Having these fantastic examples at hand, mechanical engineers developed some methods to mobilize the polyhedra. One of the engineers working on the subject is Karl Wohlhart. Wohlhart makes use of rotational type joints to expand and contract polyhedra. In 2001 he presented his method for uniform polyhedra (Fig 43) [43].

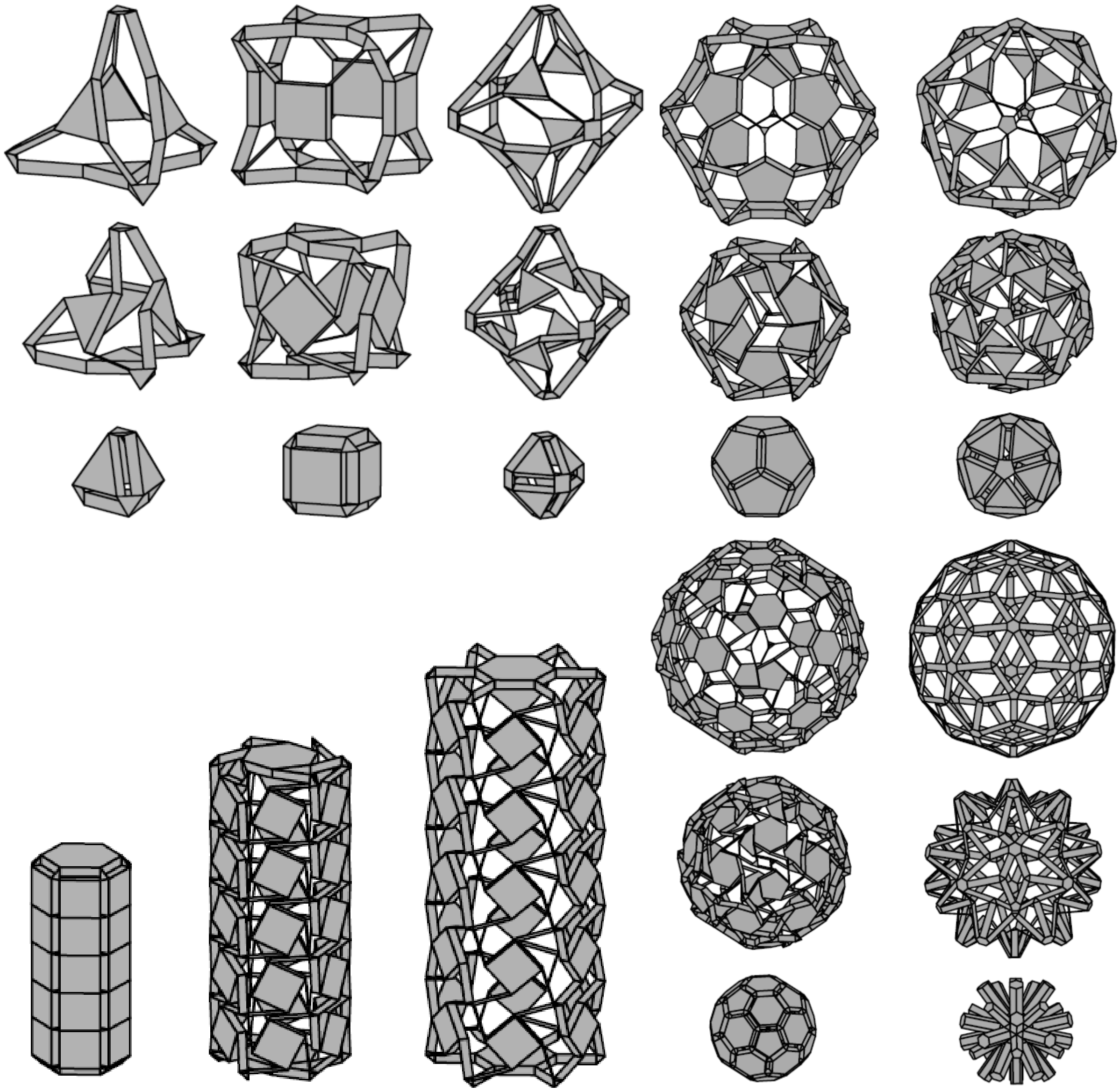

Fig 43 The five Platonic solids, a prism, the truncated icosahedron and a rhombic polyhedron mobilized by Wohlhart [43]

Later in 2004, Wohlhart applied his method to non-uniform polyhedra as well. But there is no wide range of examples for nonuniform polyhedra (Fig 44) [44].

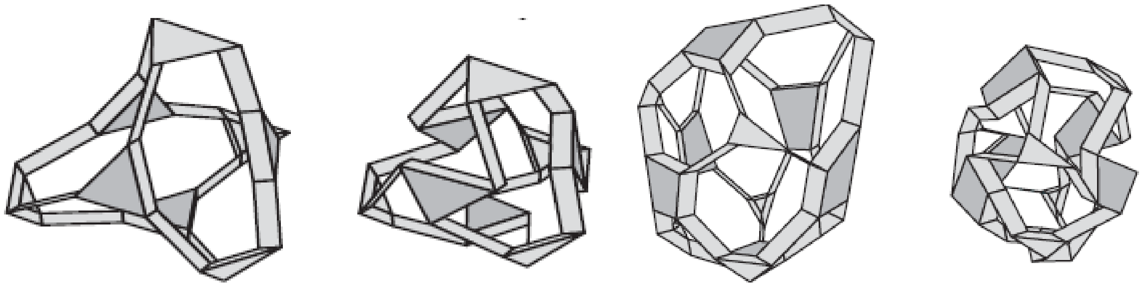

Fig 44 A non-uniform tetrahedral and a nonuniform hexahedral linkage [44]

He also gives examples for frustum pyramids and Catalan solids (duals of Archimedean solids) together with a beautiful toroidal linkage complex (Fig. 45).

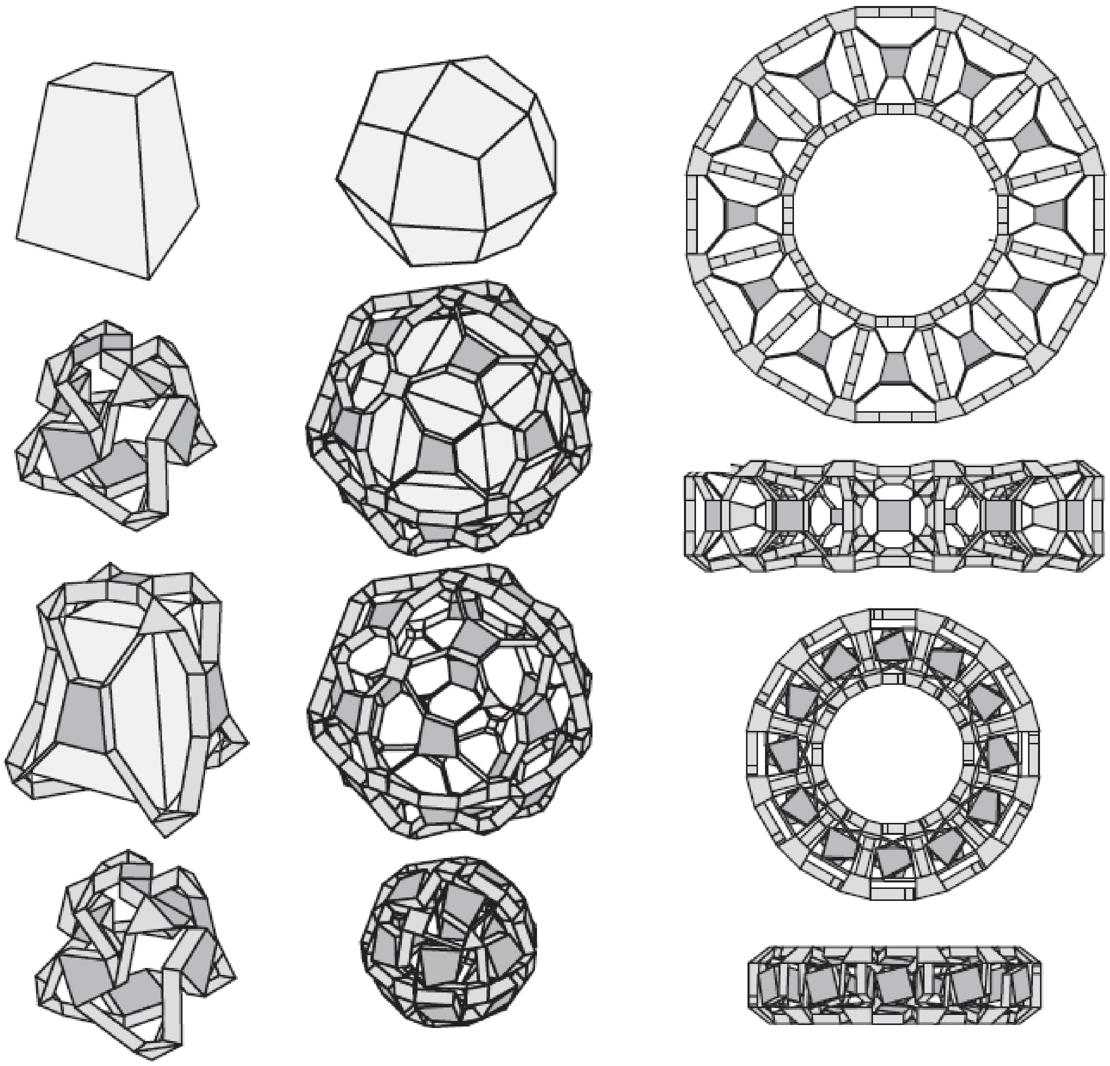

Fig 45 A frustum pyramid, a Catalan solid and a toroidal linkage complex [43]

Another method for design of polyhedral Linkages was proposed by Agrawal et al. in 2002 [45]. They placed the joints at the edges instead of the vertices, and they used prismatic [sliding] joints instead of revolute (rotational) joints. They also considered combining these linkages together to model three dimensional objects.

Recently a design methodology for a family of deployable polyhedra was proposed by Kiper et. al. [46]. Together with the Verheyen's classification of dipolygonids [47], this methodology can be counted as a rare attempt in designing polyhedral linkages systematically based on mathematical tools, rather than being individual inventions. With this methodology first, triangles are magnified, then triangles are assembled to form a polygon and finally, polygons are assembled to form polyhedra (Fig 46).

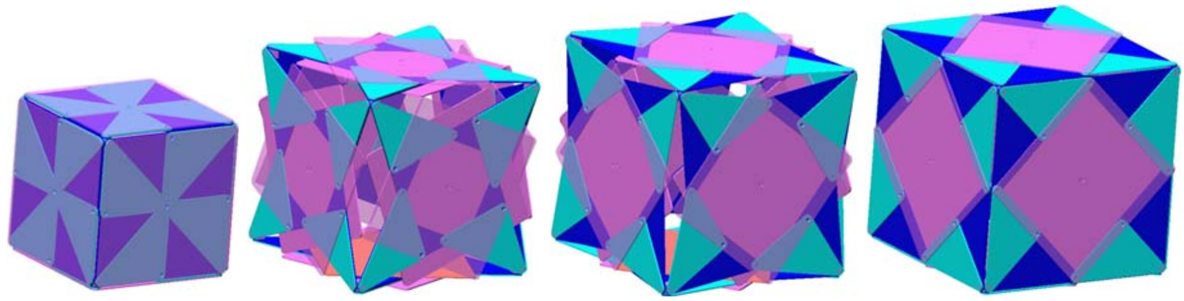

Fig 46 A cubic linkage by Kiper et. al. [46]

These are not the only designs of these inventors and researchers and some other related designs belong to Goldberg [48], Tarnai et. al. [40-41] and Gosselin et. al. [49].

SOME FUTURE WORK

Once polyhedra are mobilized, expanding-contracting models of any three dimensional object will be possible. The present problems in design of such models lack efficient unit element geometries, difficulties in actuation, malfunctioning due to friction and undesired force couples, material choice, and such. If detailed studies can be performed on solution of these problems, the practical life of humans can encounter drastic changes.

Imagine that cars can be resized to one third of its original size when parking. Or that the car can be optionally adjusted as for single person, double people or four people. Wouldn't that be a considerable ease in vehicle traffic?

Also, if such designs are made possible, the spatial studies would benefit much. Construction is very hard in space for that gravity is absent or less than the one on earth. But if contracting structure can be designed, constructions in space would be unnecessary.

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Some Useful Web Sites

http://www.korthalsaltes.com/ (Paper models of polyhedra: just print and fold)

http://aleph0.clarku.edu/~djoyce/java/elements/toc.html (Euclid, *Elements*)

http://www.zometool.com, http://www.polydron.co.uk, http://www.jovo.com (toy model producers)

http://www.ac-noumea.nc/maths/amc/polyhedr/index1_.htm (convex polyhedra animations)

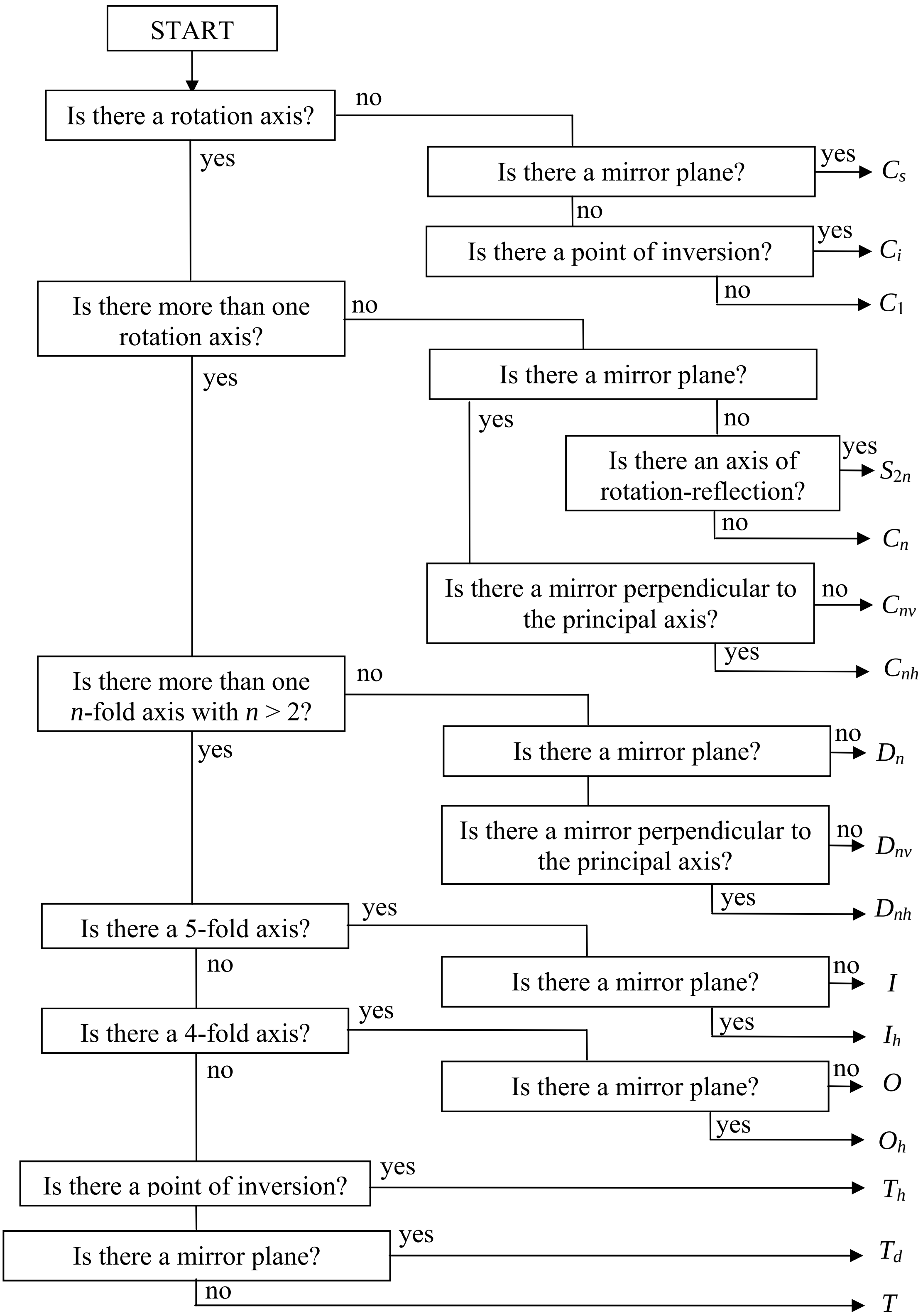
APPENDIX

SYMMETRIES OF POLYHEDRA

A rotation of $(360/n)^{\circ}$ is called an n-fold rotation. An axis of n-fold rotational symmetry is an n-fold axis. For $n = 1$, we have the *identity symmetry* (identity symmetry can also be obtained by repeated reflections).

Rotational symmetries also have subcategories: cyclic symmetries (C_n – isomorphic to the cyclic group), dihedral symmetries (D_n – isomorphic to the dihedral group), tetrahedral symmetries (T – isomorphic to the alternating group A_4), octahedral symmetries (O – isomorphic to the symmetric group S_4), icosahedral symmetries (I – isomorphic to the alternating group A_5). The details of these symmetries will not be discussed here, but it should be noted that these are the only rotational symmetries.

Reflection symmetries are subdivided into bilateral symmetry (C_s) prismatic symmetries (D_{nh}, D_{nv}, D_n, C_{nh}, C_{nv}, C_n; h is for horizontal mirror planes, v is for vertical mirror planes), compound symmetries (S_{2n}, C_i,), cubic symmetries (O_h, O, T_h, T_d, T), icosahedral symmetries (I_h, I). An asymmetric polyhedron is denoted by C_1. A polyhedron shall have one of these 17 types of symmetry. Cromwell gives an algorithm to determine the symmetry of a polyhedron [5]:


START
Is there a rotation axis?
no
yes
Is there a mirror plane?
yes
C_s
no
Is there a point of inversion?
yes
C_i
no
C_1
Is there more than one rotation axis?
no
yes
Is there a mirror plane?
yes
no
Is there an axis of rotation-reflection?
yes
S_{2n}
no
C_n
Is there a mirror perpendicular to the principal axis?
no
C_{nv}
yes
C_{nh}
Is there more than one n-fold axis with $n > 2$?
no
yes
Is there a mirror plane?
no
D_n
Is there a mirror perpendicular to the principal axis?
no
D_{nv}
yes
D_{nh}
Is there a 5-fold axis?
yes
no
Is there a mirror plane?
no
I
yes
I_h
Is there a 4-fold axis?
yes
no
Is there a mirror plane?
no
O
yes
O_h
Is there a point of inversion?
yes
T_h
Is there a mirror plane?
yes
T_d
no
T